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A GEOMETRIC FRACTIONAL MONODROMY THEOREM

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Abstract. We prove the existence of fractional monodromy for two degree of freedom integrable Hamiltonian systems with one-parameter families of curled tori under certain general conditions. We describe the action coordinates of such systems near curled tori and we show how to compute fractional monodromy using the notion of the rotation number.

1. **Introduction.** We consider two degree of freedom integrable Hamiltonian systems defined by two Poisson commuting functions F_1, F_2 on a symplectic manifold (P, ω). The map

$$F : P \to \mathbb{R}^2 : p \mapsto F(p) = (F_1(p), F_2(p)) \tag{1}$$

is called the *integral map*. We assume here, for simplicity, that fibres $\Lambda_f := F^{-1}(f)$ are compact and connected. When f is a regular value of F the fibre $F^{-1}(f)$ is a smooth two dimensional torus $\mathbb{T}^2_f$. We denote by R the set of regular values of F. When f is a critical value of F, i.e., when there are one or more points $p \in F^{-1}(f)$ where DF_p is not submersive, the fibre $F^{-1}(f)$ is either a lower dimensional torus ($\mathbb{S}^1$ or point) or a singular set (a singular algebraic variety when F is polynomial).

The question of the existence of global action-angle coordinates in integrable Hamiltonian systems was studied initially by Nekhoroshev [11]. In [6] Duistermaat classified the obstructions to the existence of global action-angle coordinates. It turns out that an integrable Hamiltonian system on a symplectic manifold P has global action-angle coordinates if and only if the 2-torus bundle $\bigcup_{f \in \mathsf{R}} \mathbb{T}^2_f \xrightarrow{F_{\mathsf{R}}} \mathsf{R}$ is trivial as a symplectic bundle [6, 10, 9]. Here F_{R} is the restriction of F to $F^{-1}(\mathsf{R})$.

The most basic obstruction to the triviality of F_{R}, and thus to the existence of global action-angle coordinates, is *Hamiltonian monodromy*, or simply *monodromy*. Monodromy characterizes the non-triviality of the bundle $\bigcup_{f \in \mathsf{R}} H_1(\mathbb{T}^2_f, \mathbb{Z}) \xrightarrow{\rho} \mathsf{R}$. The non-triviality of ρ implies the non-triviality of F_{R}. On the other hand, the triviality of ρ does imply the existence of global action coordinates but need not imply the triviality of F_{R}. In such case the system may still not have global angle

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coordinates. The latter depends on the existence of a global Lagrangian section $s : \mathsf{R} \to F^{-1}(\mathsf{R})$, for more details see [6, 10, 9].

The non-triviality of the bundle ρ can be uncovered by the parallel transport of a basis of $H_1(\mathbb{T}^2_f, \mathbb{Z}) \simeq \mathbb{Z}^2$, $f \in \mathsf{R}$ along a closed path Γ in R that starts and ends at f. Since the fibres $H_1(\mathbb{T}^2_f, \mathbb{Z})$ are discrete, such parallel transport is uniquely defined, see [6, 10, 9]. Thus we obtain the *monodromy map* which is a group homomorphism

$$\mu : \pi_1(\mathsf{R}, f) \to \mathrm{Aut}(H_1(\mathbb{T}^2_f, \mathbb{Z})), \tag{2}$$

that assigns to each homotopy class $[\Gamma]$ an automorphism of $H_1(\mathbb{T}^2_f, \mathbb{Z})$. The integrable system F in (1) has non-trivial Hamiltonian monodromy if μ in (2) is not trivial. In this case we often abuse terminology by just saying that the system has monodromy. Given a basis of $H_1(\mathbb{T}^2_f, \mathbb{Z})$ we can write the matrix M of the automorphism $\mu([\Gamma])$. M is called the monodromy matrix along Γ. Notice that an integrable system has non-trivial Hamiltonian monodromy if there is at least one closed path Γ in R such that the corresponding monodromy matrix M is not the identity.

Fractional Hamiltonian monodromy was introduced in [14] in the specific example of a two degree of freedom integrable Hamiltonian system in phase space $\mathbb{R}^4$ with coordinates $(q,p) = (q_1, q_2, p_1, p_2)$ and standard symplectic form $\omega = \mathrm{d}q_1 \wedge \mathrm{d}p_1 + \mathrm{d}q_2 \wedge \mathrm{d}p_2$ given by

$$J = F_1 = \tfrac{1}{2}(q_1^2 + p_1^2) - (q_2^2 + p_2^2), \quad F_2 = 2q_1p_1q_2 + (q_1^2 - p_1^2)p_2 + \varepsilon R(q,p)^2. \tag{3}$$

F_2 commutes with J since the cubic terms of F_2 and

$$R(q,p) = \tfrac{1}{2}(q_1^2 + p_1^2) + (q_2^2 + p_2^2),$$

commute with J. The parameter $\varepsilon > 0$ is a scaling factor. The Hamiltonian vector field X_J corresponding to J defines a locally free Hamiltonian $\mathbb{S}^1$ action on $\mathbb{R}^4_* = \mathbb{R}^4 \setminus \{0\}$. On the two dimensional subset $O = \{(q,p) \in \mathbb{R}^4_* : q_1 = p_1 = 0\}$ this $\mathbb{S}^1$ action has $\mathbb{Z}_2$ isotropy. Dynamically, this is expressed by the fact that all orbits of X_J in $\mathbb{R}^4_*$ are periodic with minimal period 2π except for those on O where the minimal period is π.

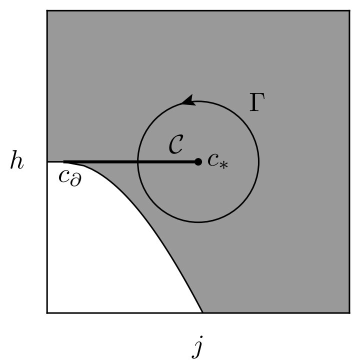
Following [7], we consider instead of $F = (J, F_2)$ in (3) the integrable system $E = (J, H)$ defined by the commuting functions

$$J, \quad H = F_2 - \varepsilon F_1^2. \tag{4}$$

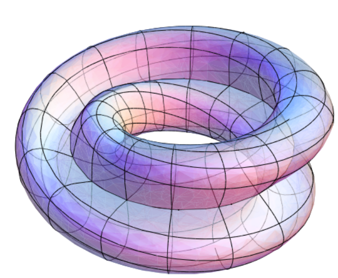
Notice that the integrable systems $F = (F_1, F_2)$ and $E = (J, H)$ are equivalent in the sense that $\psi \circ F = E$ where ψ is the diffeomorphism $(u, v) \mapsto (u, v - \varepsilon u^2)$ on $\mathbb{R}^2$. This means that F and E define the same (up to diffeomorphism) fibration of $\mathbb{R}^4$.

Remark 1. Integrable Hamiltonian systems such as (3) and (4) are called 1:(−2) *resonant systems*. The reason is that they describe the integrable fibration of the normal form approximation to perturbations $K = J + \varepsilon K_3 + \varepsilon^2 K_4 + O(\varepsilon^3)$ of the 1:(−2) resonant oscillator. Specifically, when normalized with respect to J, and truncated, the Hamiltonian K becomes $\bar{K} = J + \varepsilon \bar{K}_3 + \varepsilon^2 \bar{K}_4$ where $\{J, \bar{K}_j\} = 0$ for $j = 3, 4$. Notice that the integrable fibrations defined by the integral maps $(J, \bar{K})$ and $(J, \bar{K}_3 + \varepsilon \bar{K}_4)$ are equivalent and the systems (3) and (4) are of the latter form.

In figure 1(a) we depict the bifurcation diagram $\mathcal{BD}_E$ of E, i.e., its sets of regular and critical values. $\mathcal{BD}_E$ contains a line $\mathcal{C}$ of critical values of E that joins a point c_∂ at the boundary of $\mathcal{BD}_E$ to the origin $c_* = (0,0)$. At the point c_∂ a Hamiltonian pitchfork bifurcation takes place. $\mathcal{C}$ does not include c_∂ and c_*. For each critical value $c \in \mathcal{C}$ the fibre $\Lambda_c := E^{-1}(c)$ is a singular two dimensional surface that is called *curled torus* [14], see figure 1(b). Points at the boundary of $\mathcal{BD}_E$ lift in phase space to smooth circles and the origin c_* lifts to a degenerate singly pinched torus, see [15] for more details. The points $p \in \Lambda_c$ where $\mathrm{rank} DE_p = 1$ form a smooth circle o_c which is an orbit of X_J with period π.



(a)



(b)

FIGURE 1. (a) Bifurcation diagram $\mathcal{BD}_E$ for the 1:(−2) resonant system given by the integral map E. (b) Curled torus Λ_c.

The set of regular values R of E (light gray in figure 1(a)) is simply connected, thus the system has trivial Hamiltonian monodromy in the sense described before. The idea presented for the first time in [14] was that in order to understand the geometry of E better we can consider the set $\bar{\mathsf{R}} = \mathsf{R} \cup \mathcal{C}$ which is not simply connected (since $c_* \notin \bar{\mathsf{R}}$) and we can ask what is the geometry of the map $\bigcup_{f\in\bar{\mathsf{R}}} H_1(\mathbb{T}^2_f, \mathbb{Z}) \xrightarrow{\bar{\rho}} \bar{\mathsf{R}}$. We review here the results and ideas from [14, 15, 7, 8].

The homology group $H_1(\mathbb{T}^2_f, \mathbb{Z})$ for $f \in \mathsf{R}$ is spanned by two homology cycles that form a basis of $H_1(\mathbb{T}^2_f, \mathbb{Z})$. One of these cycles, denoted by b_f, can be chosen as the cycle generated by the flow of X_J. We denote by a_f the other basis cycle of $H_1(\mathbb{T}^2_f, \mathbb{Z})$. The homology group $H_1(\Lambda_c, \mathbb{Z})$ of the curled torus Λ_c is spanned by the cycle β_c which has as representative the period-π orbit of X_J and a second cycle that we denote by α_c. In the limit where $f \to c \in \mathcal{C}$, the cycle b_f approaches $2\beta_c$ and the cycle a_f approaches either α_c or $\alpha_c + \beta_c$ depending from which side of $\mathcal{C}$, f approaches c. It follows that $\bar{\rho}$ is not a fibre bundle. If we denote by $\mathcal{H}_f$ for $f \in \mathsf{R}$ the index-2 subgroup of $H_1(\mathbb{T}^2_f, \mathbb{Z})$ spanned by the cycles $(2a_f, b_f)$, and by $\mathcal{H}_c$ for $c \in \mathcal{C}$ the index-4 subgroup of $H_1(\Lambda_c, \mathbb{Z})$ spanned by $(2\alpha_c, 2\beta_c)$ then the map $\bigcup_{f\in\bar{\mathsf{R}}} \mathcal{H}_f \xrightarrow{r} \bar{\mathsf{R}}$ is a fibre bundle. Furthermore one can define a group homomorphism

$$\bar{\mu} : \pi_1(\bar{\mathsf{R}}, f) \to \mathrm{Aut}(\mathcal{H}_f), \tag{5}$$

by the parallel transport of homology cycles along closed paths Γ in $\bar{\mathsf{R}}$. It is necessary to consider a subgroup of the first homology group due to the fact that the only homology cycles in $H_1(\mathbb{T}^2_f, \mathbb{Z})$ that can be continued through $\mathcal{C}$ are those in the subgroup $\mathcal{H}_f$. We come back to this point in section 3.3. For a closed path Γ that goes once around c_*, see figure 1(a), there is a basis of $\mathcal{H}_f$ in which $\bar{\mu}([\Gamma])$ is given by the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{6}$$

The name "fractional monodromy" is due to the fact that if we write $\bar{\mu}([\Gamma])$ *formally* in a basis of $H_1(\mathbb{T}^2_f, \mathbb{Z})$ then its matrix is

$$\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}.$$

All existing proofs of fractional monodromy [15, 7, 8, 3, 16, 12] are given for 1:(−2) resonant systems and use in an essential way specific properties of these systems. In this paper we give sufficient conditions so that an integrable two degree of freedom Hamiltonian system with a global $\mathbb{S}^1$ action has fractional monodromy. These conditions ensure that

(1) the bifurcation diagram $\mathcal{BD}$ of the integrable system $F = (F_1, F_2)$ has a curve of critical values $\mathcal{C}$ that lift to curled tori, and that
(2) we can find a closed path Γ that intersects $\mathcal{C}$ exactly once while all other points along Γ are regular values of F.

Notice that the 1:(−2) resonant systems in (3) and (4) have both of the above properties, thus fractional monodromy for these systems is a direct consequence of our results. Furthermore, we show how to calculate the monodromy matrix by computing the variation of the rotation number along Γ.

We give a short overview of the paper. In section 2 we state our main result, Theorem 2.1, which we prove in section 3. In section 4 we study the action coordinates of the system near the curled torus Λ_c. In section 5 we show how to compute fractional monodromy through the variation of the rotation number. We close in section 6 with a discussion on the generalization of fractional monodromy to systems without a global $\mathbb{S}^1$ action.

2. **Statement of the main result.**

We consider a 2 degree of freedom integrable Hamiltonian system on a manifold P with symplectic form ω. The integral map $F = (F_1, F_2) : P \to \mathbb{R}^2$ is assumed to be smooth and proper. Here we give conditions under which we prove that F has fractional monodromy. We distinguish between local and global conditions.

2.1. **Local conditions.**

We assume that there exists a critical value $c = (c_1, c_2)$ of F such that

(L1) $\Lambda_c = F^{-1}(c)$ is connected and F_1 is non-singular on Λ_c.

Furthermore, we assume that

(L2) critical points of F on Λ_c are *transversally nondegenerate*,

i.e., if Σ is any Poincaré section of the restriction to the $F_1 = c_1$ level of the Hamiltonian flow of X_{F_1}, then the restriction of F_2 to Σ is a Morse function. This implies that F_2 is a Morse-Bott function in a neighborhood of Λ_c. The local geometry of such systems was studied in [1] the results of which we use here.

Under conditions (L1) and (L2) the critical set γ_c of F on Λ_c consists of a finite number of periodic orbits of X_{F_1} [1]. We assume that

(L3) The critical set γ_c consists of *exactly one* periodic orbit o_c of X_{F_1}.
(L4) If Σ is a two dimensional Poincaré section for o_c then $p = \Sigma \cap o_c$ is a saddle point for the restriction of F_2 to Σ, i.e., p is a corank-1 *hyperbolic singularity* of F.

Under the above conditions $\Lambda_c \setminus o_c$ contains o_c in its closure and is diffeomorphic either to the disjoint union of two cylinders (direct case) or to one cylinder (reverse case), see [1]. In the latter case Λ_c is called a *curled torus*. We finally assume that

(L5) the critical point p is a *reverse* hyperbolic singularity, i.e., Λ_c is a curled torus.

In this case there is a neighborhood U of c such that the set of critical values of F in U is a smooth curve $\mathcal{C}$ and for each $c' \in \mathcal{C} \cap U$ the fibre $\Lambda_{c'}$ is also a curled torus, see figure 2. The critical set O of F in $V = F^{-1}(U)$ is a cylinder over o_c, i.e., O is diffeomorphic to $o_c \times \mathbb{R} \simeq \mathbb{S}^1 \times \mathbb{R}$.

Remark 2. Given the local conditions (L1)-(L5), the system has a local $\mathbb{S}^1$ action in a neighborhood of Λ_c with $\mathbb{Z}^2$ isotropy on O. We discuss this point in more detail in section 3.1 but here we would like to remark that the imposed conditions exclude the cases of fractional monodromy related to higher order $m{:}(-n)$ resonances [12, 16] and fuzzy fractional monodromy [13].

2.2. **Global conditions.** We assume further that

(G1) $F = (F_1, F_2)$ is invariant under a globally defined Hamiltonian $\mathbb{S}^1$ action.

This means that there is a function J with $\{J, F_1\} = \{J, F_2\} = 0$ such that the flow of X_J is 2π periodic; the minimal period may be smaller than 2π. Condition (G1) implies that, just as in the specific example of the 1:(-2) resonant system discussed in the introduction, for each regular value $f \in \mathsf{R}$ there is a homology cycle $b_f \in H_1(\mathbb{T}^2_f, \mathbb{Z})$ that is defined by the flow of X_J. Moreover, since J is globally defined, the parallel transport of b_f from $f \in \mathsf{R}$ to any $f' \in \mathsf{R}$ along a path in R gives the cycle $b_{f'}$. The parallel transport of b_f along any closed path Γ in $\bar{\mathsf{R}} = \mathsf{R} \cup \mathcal{C}$ gives the cycle b_f. Here $\mathcal{C}$ is the one dimensional set of critical values in the image of F whose points lift to a curled torus in phase space. We assume that

(G2) there is a closed path

$$\Gamma : [0,1] \to \bar{\mathsf{R}} : t \mapsto \Gamma(t) \tag{7}$$

such that Γ transversally crosses $\mathcal{C}$ exactly once for $t = t_* \in (0,1)$ and $\Gamma(t) \in \mathsf{R}$ for $t \neq t_*$.

2.3. **Main result.** We can now state our main result.

Theorem 2.1. *Assume that an integrable Hamiltonian system $F = (F_1, F_2)$ has a curve $\mathcal{C}$ of critical values c that satisfy the local conditions (L1)-(L5) and F satisfies the global conditions (G1) and (G2). Then the cycles in $H_1(\mathbb{T}^2_f, \mathbb{Z})$ that can be continued through $\mathcal{C}$ form an index-2 subgroup $\mathcal{H}_f$ of $H_1(\mathbb{T}^2_f, \mathbb{Z})$ and there is a basis of $\mathcal{H}_{\Gamma(0)}$ in which the monodromy matrix along the path Γ satisfying condition (G2) becomes*

$$\begin{pmatrix} 1 & 0 \\ 2k-1 & 1 \end{pmatrix}, \tag{8}$$

for some $k \in \mathbb{Z}$.

Remark 3. Expressed formally in an appropriate basis of $H_1(\mathbb{T}^2_{\Gamma(0)}, \mathbb{Z})$, the monodromy transformation becomes

$$\begin{pmatrix} 1 & 0 \\ k-\frac{1}{2} & 1 \end{pmatrix}$$

thus justifying the name "fractional monodromy".

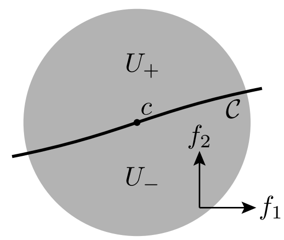


FIGURE 2. A neighborhood U of the critical value c of the integral map F that contains only critical fibres of the same type and regular fibres. The curve of critical values $\mathcal{C}$ separates U into two disjoint sets U_+ and U_-.

3. **Proof of Theorem 2.1.** We look first at an integrable system $F = (F_1, F_2)$ near a critical value c that satisfies the local conditions in section 2.1. The first step of the proof in section 3.1 is the introduction near c of a local integral map (J, H) which is equivalent to F. Then, using an appropriate Poincaré section (section 3.2) we show that only a subgroup of the first homology group can be continued through $\mathcal{C}$ (section 3.3). We complete the proof of Theorem 2.1 in section 3.4, where we take into account the global conditions on F given in section 2.2.

3.1. **The local integral map.** Let c be a critical value of the integral map F that satisfies the local conditions (L1)-(L5) in section 2.1. It follows from [1, 19] that there is an open neighborhood $U \subset \text{image}(F)$ of c such that in $V = F^{-1}(U)$ there exists a unique function J (up to an additive constant) with the following properties.

(J1) J commutes with F_1 and F_2, which implies that J factors through F, i.e., $J = J(F_1, F_2)$.
(J2) The flow of X_J is 2π periodic outside the critical set O and π periodic in O.
(J3) In O, $X_J = g(F_1)X_{F_1}$, where g is a positive function.

Thus X_J defines an $\mathbb{S}^1$ action on V with $\mathbb{Z}_2$ isotropy on O. From property (J3) of J, it follows that the map $(F_1, F_2) \to (J, F_2)$ is a diffeomorphism in U (possibly by restricting U). From now on we work with the integral map $F_{\text{local}} = (J, H)$ where $H = F_2$ in a neighborhood $V = F^{-1}(U)$ of Λ_c. The curve of critical values $\mathcal{C}$ is parameterized by $(j, h_c(j))$ and $\mathcal{C}$ separates U into the disjoint parts $U_+ = \{(j,h) \in U : h > h_c(j)\}$ and $U_- = \{(j,h) \in U : h < h_c(j)\}$, see figure 2.

Remark 4. The function J is defined locally in V. If the system has a global Hamiltonian $\mathbb{S}^1$ action generated by the Hamiltonian vector field associated to a function J' and the latter factors through F, then J' satisfies the properties (J1)-(J3) and $J' - J$ is constant in V.

3.2. **Poincaré section.** Fix a critical value $(j_c, h_c(j_c))$ of the integral map F_{local} and consider the set $V_{j_c} = J^{-1}(j_c) \cap V \subset P$.

Lemma 1. *There exists in V_{j_c} a local smooth two dimensional Poincaré section Σ_{local} to the critical orbit o_c such that the return map is the time π flow of X_J.*

Proof. Consider an arbitrary section Σ'_{local}. The point p on $o_c \cap \Sigma'_{\text{local}}$ returns to Σ'_{local} along the flow of X_J after time π (since o_c is a periodic orbit of X_J with period π). Using the implicit function theorem we conclude that the return time $t(x)$ for x

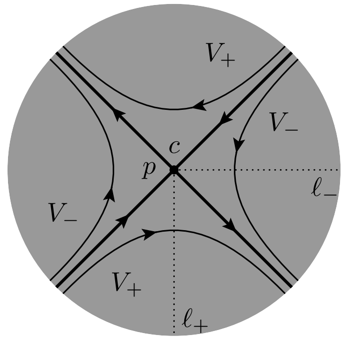


FIGURE 3. Level curves of H on the local Poincaré section Σ_{local}.

in an open neighborhood W of p in Σ'_{local} is a smooth function $t : W \to \mathbb{R}$. Denote by $\Phi_J(t, x)$, $t \in \mathbb{R}$, $x \in P$, the time t flow of the Hamiltonian vector field X_J and let $x' = \Phi_J(t(x), x)$ for $x \in W$. Then $t(x') + t(x) = 2\pi$. We can construct a local Poincaré section Σ_{local} with the property that the return time is π by considering the map $g : W \to P$ given by

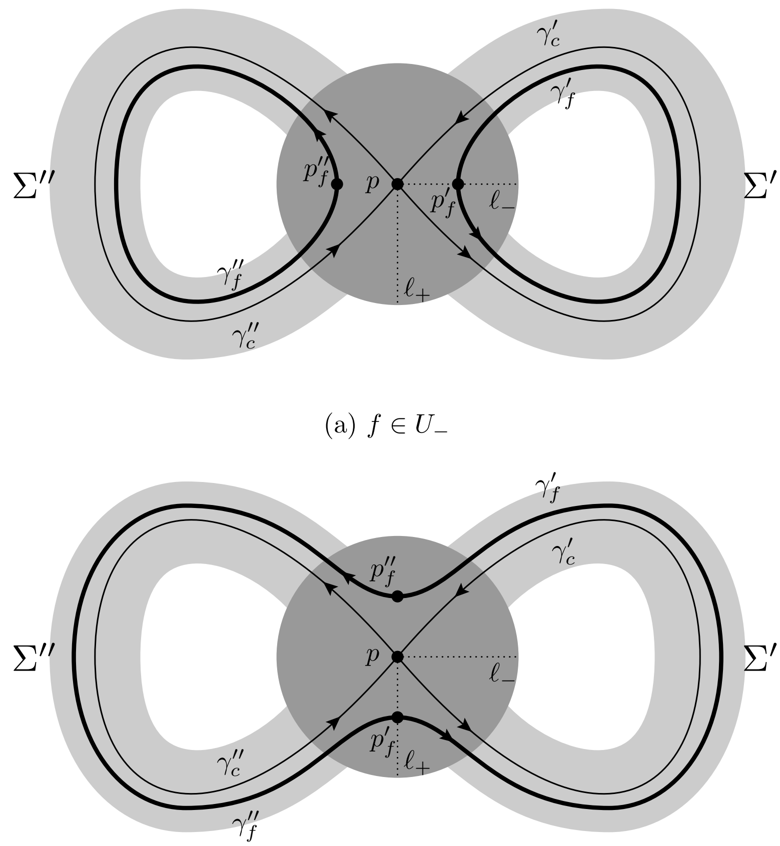
$$g(x) = \Phi_J(s(x), x)$$

where $s(x) = (t(x) - \pi)/2$. Then points on $\Sigma_{\mathrm{local}} = g(W)$ return to Σ_{local} after time π. □

The structure of the level sets of H on Σ_{local} follows from the local conditions in section 2.1. These level sets are depicted in figure 3. The intersection of Λ_c with Σ_{local} consists of the point p and the curves w^s and w^u that intersect transversally at p. If we consider the dynamics induced by X_H when projected along the direction of X_J on Σ_{local} then the curves w^s and w^u can be characterized as parts of the local stable and unstable manifolds respectively of the unstable fixed point p. In order to extend this local section along Λ_c we consider any smooth closed curve γ'_c on $\Lambda_c \setminus o_c$ that connects p through a branch of w^u to p through a branch of w^s and such that X_J is transversal to γ'_c. Further, denote by γ''_c the image of γ'_c under $\Phi_J(\pi, -)$. Then γ''_c connects the two remaining branches of w^s and w^u. We represent the situation schematically in figure 4. Notice that the curves γ'_c and γ''_c can not intersect (except at p) since if they intersect at a point p', then p' is a fixed point of the Poincaré return map, i.e., corresponds to an orbit of X_J with $\mathbb{Z}_2$ isotropy. From our assumptions the fibre Λ_c contains only one such orbit.

Then we extend the local Poincaré section Σ_{local} smoothly along γ'_c to obtain a surface Σ' that contains γ_c and such that X_J is transversal to Σ'. We map Σ' by the symplectic map $\Phi_J(\pi, -)$ in order to obtain a second part $\Sigma'' = \Phi_J(\pi, \Sigma)$ and we define the 'global' section $\Sigma = \Sigma_{\mathrm{local}} \cup \Sigma' \cup \Sigma''$, see figure 4. Denote by λ_f the intersection of the fibre $\Lambda_f = F^{-1}_{\mathrm{local}}(f)$, $f = (j, h)$ with Σ. Depending on the choice of γ_c on Λ_c there are two possibilities. Either for $f \in U_-$, λ_f consists of two components and for $f \in U_+$, λ_f consists of one component, or the opposite. In order to fix notation and without loss in generality we consider the first case, see figure 4.

3.3. **Construction and continuation of homology cycles.** For the construction of a basis of the first homology group $H_1(\Lambda_f, \mathbb{Z})$ of fibers near a curled torus Λ_c



(b) $f \in U_+$

FIGURE 4. Intersections λ_f of fibers Λ_f with the Poincaré section Σ which consists of Σ_{local}, shown here in dark gray, and the parts Σ' and Σ'', shown in light gray. The intersection of critical fiber Λ_c with Σ is the figure-8 thin curve that consists of γ'_c and γ''_c. (a) For $f \in U_-$ the intersection λ_f (thick curve) of the fibre Λ_f with Σ consists of two closed curves γ'_f and γ''_f. The curve γ'_f starts and ends at the point $p'_f \in \ell_-$. (b) For $f \in U_+$ the intersection λ_f (thick curve) is a single closed curve that starts and ends at the point $p'_f \in \ell_+$. In this case λ_f consists of the open curves γ'_f and γ''_f.

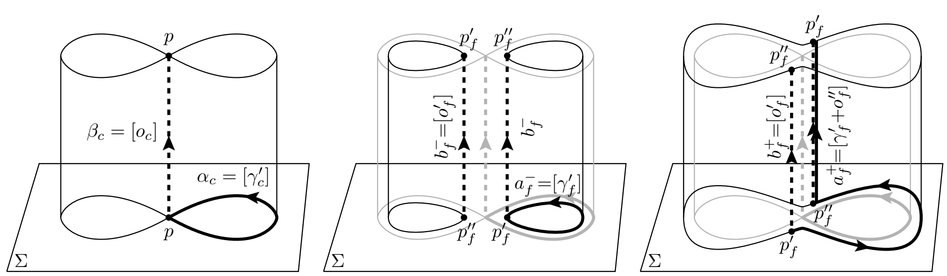
we use the Poincaré surface of section Σ constructed in section 3.2. The continuation of homology cycles through the curve $\mathcal{C}$ of critical values of F is also described in detail in [8].

Consider two smooth curves $\ell_\pm$ on $V_\pm \cap \Sigma_{\text{local}}$ that join (not smoothly) at $p \in o_c$, see figures 3 and 4. Each curve $\ell_\pm$ intersects transversally the level sets λ_f. We denote by ℓ the union of $\ell_\pm$ and we let $p'_f = \lambda_f \cap \ell$ and $p''_f = \Phi_J(\pi, p'_f)$.

The homology of the singular fibre Λ_c is generated by two elements α_c and β_c, see figure 5(a). A representative of α_c is the closed curve γ'_c on Σ, constructed in section 3.2 and shown also in figure 4, that connects p to itself. A representative of β_c is o_c.

For $f = (j_c, h) \in U_-$ we consider the curve γ'_f on Σ that starts and ends at the point p'_f, see figure 4(a). This curve represents a homology cycle a_f^-, see figure 5(b). The second element of the basis of $H_1(\Lambda_f, \mathbb{Z})$ is denoted by b_f^- and is represented by the X_J orbit o'_f that starts and ends at p'_f. As $f \in U_-$ approaches c, i.e., as $h \to h_c(j_c)^-$, the point p'_f on ℓ_- approaches p. Furthermore, the representative curve γ'_f approaches γ_c and thus a_f^- approaches α. As p'_f, $f \in U_-$ approaches p along ℓ_-, the curve o'_f approaches $2o_c$, thus the cycle b_f^- approaches $2\beta_c$.

For $f = (j_c, h) \in U_+$, λ_f is a single closed curve on Λ_f, see figure 4(b). Given the point p'_f a basis of the first homology group $H_1(\Lambda_f, \mathbb{Z})$ is given by the cycles a_f^+ and b_f^+ depicted in figure 5(c). The cycle b_f^+ is represented by the closed orbit o'_f of the flow of X_J that starts and ends at p'_f. The cycle a_f^+ is represented by the sum of the curve $\gamma'_f \subset \lambda_f$ that starts at p'_f and ends at $p''_f = \Phi_J(\pi, p'_f)$ (see figure 4(b)), and the curve o''_f that is defined by the X_J flow and connects p''_f to p'_f. The closed curve $\gamma_f = \gamma'_f + o''_f$ represents the cycle b_f^+. As $f \in U_+$ approaches c, i.e., as $h \to h_c(j_c)^+$, the point p'_f on ℓ_+ approaches p. Furthermore, the representative curve γ_f approaches $\gamma'_c + o_c$ and thus a_f^+ approaches $\alpha_c + \beta_c$. As p'_f, $f \in U_+$ approaches p along ℓ_+, the curve o'_f approaches $2o_c$, thus the cycle b_f^+ approaches $2\beta_c$.



(a) (b) (c)

FIGURE 5. Limits of homology cycles near the singular fibre Λ_c. In this representation of the fibres the vertical lines represent orbits of X_J. The lower and upper parts of these figures are identified after rotation of the upper part by π around the point p. (a) Basis (α_c, β_c) of $H_1(\Lambda_c, \mathbb{Z})$. (b) Basis (a_f^-, b_f^-) of $H_1(\Lambda_f, \mathbb{Z})$, $f \in U_-$. (c) Basis (a_f^+, b_f^+) of $H_1(\Lambda_f, \mathbb{Z})$, $f \in U_+$. Notice that $(a_f^-, b_f^-) \to (\alpha_c, 2\beta_c)$ and $(a_f^+, b_f^+) \to (\alpha_c + \beta_c, 2\beta_c)$.

Therefore we obtain that the limit of the basis (a_f^-, b_f^-) as $f \in U_-$ approaches $\mathcal{C}$ is $(\alpha_c, 2\beta_c)$ while for $f \in U_+$ the basis (a_f^+, b_f^+) approaches $(\alpha_c + \beta_c, 2\beta_c)$. The two bases $(\alpha_c, 2\beta_c)$ and $(\alpha_c + \beta_c, 2\beta_c)$ span different subgroups of $H_1(\Lambda_c, \mathbb{Z})$. This means that it is not possible to connect the homology bases (a_f^+, b_f^+) and (a_f^-, b_f^-) continuously through $\mathcal{C}$. For this reason we consider, following [8, 7, 14, 15], the homology bases $(2a_f^-, b_f^-)$ and $(2a_f^+, b_f^+)$. As f approaches c, these two bases approach the bases $(2\alpha_c, 2\beta_c)$ and $(2\alpha_c + 2\beta_c, 2\beta_c)$ respectively, that span the same index-4 subgroup $\mathcal{H}_c$ of $H_1(\Lambda_c, \mathbb{Z})$. Thus elements of the subgroups generated by $(2a_f^\pm, b_f^\pm)$ can be continued through the line of critical values $\mathcal{C}$. We denote by $\mathcal{H}_f$ the subgroup of $H_1(\Lambda_f, \mathbb{Z})$ that is spanned by $(2a_f^\pm, b_f^\pm)$ for $f \in U_\pm$. Notice that the basis $(2a_f^+, b_f^+)$ is continued through $\mathcal{C}$ to the basis $(2a_f^- + b_f^-, b_f^-)$.

3.4. **Fractional monodromy.** In this final part of the proof we take into account the global conditions for F in section 2.2.

Consider a closed curve $\Gamma : [0,1] \to \mathbb{R}^2$ in the image of F such that $\Gamma(t)$ is a regular value of F for all $t \in [0,1]$ except a $t_* \in (0,1)$ for which $\Gamma(t_*) \in \mathcal{C}$. Moreover we assume that Γ intersects $\mathcal{C}$ transversally, $\Gamma(0) = \Gamma(1) = f_0 \in U_-$ and that there is an $\epsilon > 0$ such that for $t_* - \epsilon < t < t_*$, $\Gamma(t) \in U_+$, while for $t_* < t \le 1$, $\Gamma(t) \in U_-$.

Consider the basis $(a_{f_0}^-, b_{f_0}^-)$ of $H_1(\Lambda_{f_0}, \mathbb{Z})$ defined in section 3.3. The parallel transport of this basis along Γ to a point $f_t = \Gamma(t) \in U_+$ with $t \in (t_* - \epsilon, t_*)$ gives a basis (c, d) of $H_1(\Lambda_{f_t}, \mathbb{Z})$. The latter basis is related to the basis $(a_{f_t}^+, b_{f_t}^+)$, constructed in section 3.3, by

$$\begin{pmatrix} c \\ d \end{pmatrix} = (N^{-1})^{\mathrm{t}} \begin{pmatrix} a_{f_t}^+ \\ b_{f_t}^+ \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}, \tag{9}$$

for some $k \in \mathbb{Z}$. In general, the matrix N that describes the parallel transport from f_0 to f_t is an arbitrary element of $\mathrm{GL}(2, \mathbb{Z})$.[1] Here the existence of the global $\mathbb{S}^1$ action generated by the flow of X_J implies that parallel transport of the cycle $b_{f_0}^- \in H_1(\Lambda_{f_0}, \mathbb{Z})$ gives the cycle $b_{f_t}^+ \in H_1(\Lambda_{f_t}, \mathbb{Z})$. This determines the special form of N.

The basis $(2a_{f_0}^-, b_{f_0}^-)$ of the subgroup $\mathcal{H}_{f_0}$ that can be continued through $\mathcal{C}$, is parallel transported to $(2c, d) = (2a_{f_t}^+ - 2kb_{f_t}^+, b_{f_t}^+)$. Recall that when $(2a_{f_t}^+, b_{f_t}^+)$ is continued through $\mathcal{C}$ it is mapped to $(2a_{f_0}^- + b_{f_0}^-, b_{f_0}^-)$. Thus $(2c, d)$ is continued to $(2a_{f_0}^- - (2k-1)b_{f_0}^-, b_{f_0}^-)$. This means that the monodromy matrix along Γ when expressed in the basis $(2a_{f_0}^-, b_{f_0}^-)$ is

$$M = \begin{pmatrix} 1 & 0 \\ 2k-1 & 1 \end{pmatrix}.$$

This concludes the proof of Theorem 2.1.

4. **Action coordinates near Λ_c.** Recall from section 3.1 that if c is a critical value of the integral map F that satisfies the local conditions in section 2.1 there is an open neighborhood V of Λ_c in which we can define a local action coordinate J and a corresponding local integral map F_{local}. Since F is an integrable Hamiltonian system a second action coordinate I exists in the preimages $V_\pm = F_{\mathrm{local}}^{-1}(U_\pm)$ of the simply connected sets $U_\pm$ of regular values of F_{local}. In this section we establish the relation between I in V_+ and I in V_-. This relation is a straightforward consequence of the relation between the homology cycles a_f^+ and a_f^- constructed in section 3.3. The second action coordinate I is a function in $V_\pm$ which is constant on the fibres Λ_f and we denote its value on Λ_f by $I(f)$.

[1] When the base of the fibration is an orientable manifold then $N \in \mathrm{SL}(2, \mathbb{Z})$ [4].

Lemma 2. *There is a second action coordinate I defined in $U_\pm$ such that*

$$I(f) = \begin{cases} A(f) + \frac{1}{2}J(f), & \textit{for } f \in U_+ \\ A(f), & \textit{for } f \in U_- \end{cases} \tag{10}$$

where $f = (J(p), H(p))$ and $A : U \to \mathbb{R}$ is smooth in $U \setminus \mathcal{C}$ and continuous at $\mathcal{C}$.

Proof. The symplectic form ω is exact in the open neighborhood V of Λ_c (possibly by taking a subset of the original V, see Lemma 1.9 in [1]), i.e., $\omega = \mathrm{d}\vartheta$ for an 1-form ϑ. This means that the second action I in $V_\pm$ can be defined by integrating ϑ over any representative of the homology cycles $a_f^\pm$. Recall that by construction, J is obtained by integrating ϑ over a representative of $b_f^\pm$, thus

$$J(f) = \frac{1}{2\pi} \int_{o'_f} \vartheta,$$

where o'_f is defined in section 3.3. For $f \in U$ let

$$A(f) = \frac{1}{4\pi} \int_{\gamma_f} \vartheta = \frac{1}{2\pi} \int_{\gamma'_f} \vartheta = \frac{1}{2\pi} \int_{\gamma''_f} \vartheta, \tag{11}$$

where $\gamma_f = \gamma'_f + \gamma''_f$, see section 3.3, and where we used the fact that $\int_{\gamma'_f} \vartheta = \int_{\gamma''_f} \vartheta$ since $\gamma''_f = \Phi_J(\pi, \gamma'_f)$ and $\Phi_J(\pi, -)$ is symplectic. The function $A(f)$ is smooth in U_+ and U_-. This follows from the fact that we integrate the smooth 1-form ϑ over the path γ_f that in $U_\pm$ depends smoothly on f. At $\mathcal{C}$, A is only continuous.

For $f \in U_-$ we define $I(f)$ by integrating ϑ over a representative of b_f^-:

$$I(f) = \frac{1}{2\pi} \int_{\gamma'_f} \vartheta = \frac{1}{4\pi} \Big(\int_{\gamma'_f} \vartheta + \int_{\gamma''_f} \vartheta \Big) = A(f),$$

where we used again here that $\int_{\gamma'_f} \vartheta = \int_{\gamma''_f} \vartheta$.

For $f \in U_+$ we define $I(f)$ by integrating ϑ over a representative of b_f^+:

$$I(f) = \frac{1}{2\pi} \int_{\gamma'_f + o''_f} \vartheta,$$

where o''_f is defined in section 3.3. Then we have

$$2I(f) = \frac{1}{2\pi} \int_{2(\gamma'_f + o''_f)} \vartheta = \frac{1}{2\pi} \int_{2\gamma'_f + o'_f} \vartheta = 2A(f) + J(f).$$

Thus, for $f \in U_+$ we obtain

$$I(f) = A(f) + \tfrac{1}{2}J(f).$$

□

Remark 5. In section 3 we saw that we can consider the subgroup $\mathcal{H}_f$ of $H_1(\mathbb{T}_f^2, \mathbb{Z})$ which is spanned by $(2a_f, b_f)$ and the elements of which can be continued through $\mathcal{C}$. At a regular fibre $\Lambda_f = \mathbb{T}_f^2$ the vector fields corresponding to the action coordinates form a discrete group isomorphic to $H_1(\mathbb{T}_f^2, \mathbb{Z}) \simeq \mathbb{Z}^2$. This discrete group is called the *period lattice*. In the case of fractional monodromy the period lattice is not defined at $\mathcal{C}$ because the Hamiltonian vector field X_I of the second action I defined in (10) blows up at $\mathcal{C}$. Any other choice of action I' such that $(X_J, X_{I'})$ is a basis of the period lattice presents the same problem since $X_{I'} = X_I + nX_J$, for some $n \in \mathbb{Z}$.

5. **The rotation number near Λ_c and computation of fractional monodromy.** Monodromy (integer or fractional) is a topological property of the fibration of the phase space defined by the integral map F. Nevertheless, monodromy can be calculated using the dynamics defined by the vector fields X_{F_1} and X_{F_2} in particular using the dynamical notions of first return time and rotation number. In this section we show how the fractional monodromy matrix along a closed path Γ can be computed through the total variation of the rotation number along Γ. Notice that two equivalent integral maps F and $\psi \circ F$, where $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism, define the same fibration of the phase space but induce different dynamics on the invariant tori and thus define different rotation numbers. For integer monodromy, although the rotation number depends on the dynamics its variation along a path Γ depends only on the geometry of the fibration. This remains true for fractional monodromy provided that, F is chosen in such a way so that the rotation number does not diverge when Γ crosses the curve $\mathcal{C}$ of critical values of F so that the variation is well defined.

We give now the definition of rotation number and first return time in a system with global $\mathbb{S}^1$ action generated by the flow of X_J. Consider any point p on $\Lambda_f = \mathbb{T}^2_f$ where $f = (j, h)$ is a regular value of (J, H), the closed orbit $o_f(p) = \{\Phi_J(t, p) : t \in [0, 2\pi]\}$, and the orbit of the flow of X_H that starts at p and intersects $o_f(p)$ after time $T(f)$ at a point p'. The time $T(f)$ is the *first return time*, it is independent of the point p and it is a locally smooth function of $f \in \mathsf{R}$. Thus this construction defines a map $\varphi : p \mapsto p' = \Phi_H(T(f), p)$ for all $p \in \mathbb{T}^2_f$. As f approaches $\mathcal{C}$, the first return time diverges because of the hyperbolic dynamics on Λ_c, $c \in \mathcal{C}$. The *rotation number* $\Theta(f) \in [0, 2\pi) \subset \mathbb{R}$ is defined as the minimum positive time that it takes to go from p to p' along the flow of X_J and does not depend on the choice of $p \in \Lambda_f$. The map $\mathsf{R} \to \mathbb{S}^1 : f \mapsto [\Theta(f)]$ is locally smooth, where $[x] \in \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ is the equivalence class of $x \in \mathbb{R}$ with respect to the equivalence relation $x \sim x + 2k\pi$, $k \in \mathbb{Z}$.

Remark 6. The restriction to any X_J orbit o_f of the map $\varphi : p \mapsto p'$ described above is a *solid rotation* by angle $\Theta(f)$ of the circle o_f, see [5].

In an open neighborhood of Λ_f where $\Theta(f)$ and $T(f)$ are smooth define the smooth Hamiltonian vector field

$$X_S(p) = -\tfrac{1}{2\pi}\Theta(j,h)X_J(p) + \tfrac{1}{2\pi}T(j,h)X_H(p), \tag{12}$$

where $f = (j, h) = (J(p), H(p))$. X_S is periodic with period 2π, thus S is an action coordinate. Furthermore the homology classes of the integral curves of X_J and X_S on the fibre Λ_f form a basis of $H_1(\Lambda_f, \mathbb{Z})$. From (12) we obtain

$$\Theta(j,h) = -2\pi\frac{\partial S(j,h)}{\partial j}, \quad T(j,h) = 2\pi\frac{\partial S(j,h)}{\partial h}. \tag{13}$$

S is obtained defined by integrating equation (13).

Recall that in section 4 we defined an action I such that the flows of (X_J, X_I) generate a basis of $H_1(\Lambda_f, \mathbb{Z})$. Thus we have that $X_S = X_I + n_\pm X_J$, $n_\pm \in \mathbb{Z}$ for $f \in U_\pm$. Then from (13) we obtain that for $(j, h) \in U_\pm$

$$\Theta(j,h) = -2\pi\frac{\partial I(j,h)}{\partial j} - 2\pi n_\pm. \tag{14}$$

Using (10) we obtain that for $(j,h) \in U \setminus \mathcal{C}$

$$T(j,h) = \frac{\partial A(j,h)}{\partial h} \tag{15}$$

and

$$\Theta(j,h) = \begin{cases} -2\pi\,\partial A(j,h)/\partial j - 2\pi n_-, & \text{for } (j,h) \in U_-, \\ -2\pi\,\partial A(j,h)/\partial j - \pi - 2\pi n_+, & \text{for } (j,h) \in U_+. \end{cases} \tag{16}$$

Thus in order to understand the behavior of the rotation number near $\mathcal{C}$ we need to understand $A(j,h)$. We have the following lemma.

Lemma 3. *$\partial A(j,h)/\partial j$ is continuous at $\mathcal{C}$, if and only if $H|_{\mathcal{C}}$ is constant. In particular, if $H|_{\mathcal{C}}$ is not constant then $\partial A(j,h)/\partial j$ blows up at $\mathcal{C}$.*

Proof. Assume that $\lim_{h\to h_c(j)} \partial A(j,h)/\partial j = \partial A(j,h_c)/\partial j$. Differentiating $A(j,H(j,a)) = a$ gives

$$\frac{\partial H(j,a)}{\partial j} = -\Big(\frac{\partial A(j,h)}{\partial h}\Big)^{-1}\frac{\partial A(j,h)}{\partial j} = -2\pi T(j,h)^{-1}\frac{\partial A(j,h)}{\partial j},$$

and

$$\frac{\partial H(j,a)}{\partial a} = \Big(\frac{\partial A(j,h)}{\partial h}\Big)^{-1} = 2\pi T(j,h)^{-1}.$$

In the limit where $h \to h_c(j)$ (or equivalently $a \to a_c(j) = A(j,h_c(j))$), the first return time $T(j,h)$ diverges, thus

$$\lim_{h\to h_c(j)} \frac{\partial H(j,a)}{\partial j} = 0, \quad \text{and} \quad \lim_{h\to h_c(j)} \frac{\partial H(j,a)}{\partial a} = 0.$$

Working for j in a compact interval $[j_1,j_2]$ we obtain that $\lim_{h\to h_c} H(j,a)$ exists uniformly in j. Thus we can interchange limits to obtain

$$\lim_{h\to h_c(j)} \frac{\partial H(j,a)}{\partial j} = 0, \quad \text{and} \quad \lim_{h\to h_c(j)} \frac{\partial H(j,a)}{\partial a} = \frac{\partial H(j,a_c)}{\partial a} = 0.$$

Thus

$$DH(j,a_c(j)) = 0,$$

which implies that H is constant along the curve $\mathcal{C}$ parameterized here by $a_c(j)$.

Now assume that $h_c(j) = 0$ along $\mathcal{C}$. For h in a compact interval $[-h_0,h_0]$ the derivative $\partial A(j,h)/\partial j$ exists uniformly in h and we obtain

$$\lim_{h\to h_c} \frac{\partial A(j,h)}{\partial j} = \frac{\partial}{\partial j}\lim_{h\to h_c} A(j,h) = \frac{\partial A(j,h_c)}{\partial j}.$$

□

Lemma 3 shows that in order to compute the total variation of the rotation number along a close path Γ we should consider a Hamiltonian H with constant value $H|_{\mathcal{C}}$. This consideration played an essential role in [7]. Since for an arbitrary Hamiltonian H the curve $\mathcal{C}$ is a graph over j, i.e., it is parameterized as $(j,h_c(j))$ where $j \in (j_1,j_2) \subset \mathbb{R}$, we obtain the required Hamiltonian function H_0 by extending h_c to a smooth function w over $\mathbb{R}$ and defining $H_0 = H - w(J)$. Thus we have an integral map $F_0 = (J,H_0)$ which is equivalent to F since the map $(J,H) \mapsto (J,H_0)$ is a diffeomorphism.

In order to compute the fractional monodromy we consider the integral map F_0 and a closed path $\Gamma : [0,1] \to \bar{\mathsf{R}}$ satisfying the conditions in section 2.2 and where in particular $\Gamma(t_*) \in \mathcal{C}$ for some $t_* \in (0,1)$. Along Γ consider the function

$\tilde{\Theta} : [0,1] \to \mathbb{R}$ defined as the unique continuous function on $[0,1]$ that satisfies $\tilde{\Theta}(0) = 2\Theta(\Gamma(0))$ and $\tilde{\Theta}(t) = 2\Theta(\Gamma(t)) \mod 2\pi$ for all $t \in [0,1]$. Then the off-diagonal entry $2k-1$ in the monodromy matrix (8) is given by

$$\lim_{t\to t_*^+} \tilde{\Theta}(t) - \lim_{t\to t_*^-} \tilde{\Theta}(t) = -2\pi(2k-1).$$

6. **Discussion.** We have showed that in 2 degree of freedom integrable Hamiltonian systems with curled tori and a global $\mathbb{S}^1$ action we have fractional monodromy. Furthermore, we showed how to compute the monodromy through the rotation number. We discuss now two further aspects of this problem: the possibility of generalizing fractional monodromy to systems without a global $\mathbb{S}^1$ action, and the relation of the integer k that appears in the fractional monodromy matrix to the type of critical values encircled by the closed path Γ.

6.1. **Fractional monodromy for systems with no global $\mathbb{S}^1$ action.** The condition that the system has a global $\mathbb{S}^1$ action plays a role only in the last part of the proof of Theorem 2.1 where we pass from the local to the global description. This condition gives the specific form of the transformation matrix N in (9). If a system does not have a global $\mathbb{S}^1$ action the situation can be more complicated.[2] In particular, it is possible that under a more general transformation matrix $N = (n_{ij}) \in \mathrm{GL}(2,\mathbb{Z})$, $i = 1,2$, $j = 1,2$, the basis $(2a^-_{f_0}, b^-_{f_0})$ of $\mathcal{H}_{f_0}$ is parallel transported to elements of $H_1(\Lambda_{f_t},\mathbb{Z})$ that do not belong in $\mathcal{H}_{f_t}$ and thus can not be continued through $\mathcal{C}$. This occurs when n_{21} is an odd number.

A way to overcome this problem is to consider the index-4 subgroup $\bar{\mathcal{H}}_{f_0}$ generated by $(2a^-_{f_0}, 2b^-_{f_0})$. Notice that $\bar{\mathcal{H}}_{f_0}$ is a subgroup of $\mathcal{H}_{f_0}$ thus all its elements can be continued through $\mathcal{C}$. The matrix N expressed in the bases $(2a^-_{f_0}, 2b^-_{f_0})$ and $(2a^+_{f_t}, 2b^+_{f_t})$ remains the same. Since $N \in \mathrm{GL}(2,\mathbb{Z})$ the subgroup $\bar{\mathcal{H}}_{f_0}$ is mapped onto $\bar{\mathcal{H}}_{f_t}$ generated by $(2a^+_{f_t}, 2b^+_{f_t})$. But if then we continue the basis $(2a^+_{f_t}, 2b^+_{f_t})$ through $\mathcal{C}$ we obtain $(2a^-_{f_0} + b^-_{f_0}, 2b^-_{f_0})$ which spans a subgroup $\bar{\bar{\mathcal{H}}}_{f_0}$ of $\mathcal{H}_{f_0}$ different from $\bar{\mathcal{H}}_{f_0}$. Notice that this remains true in systems with a global $\mathbb{S}^1$ action. Thus we see that in the case of an integrable system with no global $\mathbb{S}^1$ action we cannot define a notion of fractional monodromy that resembles the standard notion of Hamiltonian monodromy.

Remark 7. The previous discussion shows that in integrable systems with one-parameter families of curled tori, with or without a global $\mathbb{S}^1$ action, we can characterize fractional monodromy as a linear transformation between two index-4 subgroups of $H_1(\Lambda_{f_0},\mathbb{Z})$.

6.2. **Critical values and the fractional monodromy matrix.** The geometric monodromy theorems in [18, 17, 2] relate the form of the monodromy matrix along a closed path Γ to the number and type of critical values encircled by Γ. In particular, if Γ goes around k_{ff} focus-focus points then the monodromy matrix along Γ is

$$\begin{pmatrix} 1 & 0 \\ k_{\mathrm{ff}} & 1 \end{pmatrix}.$$

[2] We do not have specific examples of systems that satisfy all the assumptions of Theorem 2.1 except the assumption of a global $\mathbb{S}^1$ action but we do not know of any obstructions that would preclude the existence of such systems.

In the case of fractional monodromy we found that the fractional monodromy matrix is, formally,

$$\begin{pmatrix} 1 & 0 \\ k-\frac{1}{2} & 1 \end{pmatrix},$$

for *some* $k \in \mathbb{Z}$ but we did not specify the value of k or how it is related to the type of singularities encircled by Γ. In the specific example of the $1{:}(-2)$ resonant system (4) discussed in section 1 we have $k = 1$, cf. the fractional monodromy matrix in (6). Our conjecture is that, in this example, $k = 1$ is determined by the type of the degenerate singularity of F at the origin. Furthermore, the existence of a global $\mathbb{S}^1$ action and the orientability of the base space of the fibration imply that we can define the *sign* of monodromy [4]. From this we conclude that if Γ encircles the same degenerate singularity as in the $1{:}(-2)$ resonant system and also k_{ff} focus-focus points then the off-diagonal element in the formal fractional monodromy matrix is $k_{\mathrm{ff}} + \frac{1}{2}$. The precise way on which the fractional monodromy matrix depends on the types of singularities of F encircled by Γ is one of the most interesting directions toward which this subject can be developed.

6.3. **Final remarks.** We would like to close with a final remark and motivation for further work. Many of the constructions that were used in this paper become more natural when lifted to a covering space. Furthermore, such approach makes easier the study of higher order resonances and fuzzy fractional monodromy. We are using the notion of covering maps in order to study fractional monodromy in a forthcoming paper.

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