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Fractional monodromy in the 1:−2 resonance

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Abstract

We give an analytic proof of the fractional monodromy theorem for the 1:−2 oscillator system with $\mathbf{S}^1$ symmetry formulated by N.N. Nekhoroshev, D.A. Sadovskií, and B.I. Zhilinskií in C. R. Acad. Sci. Paris, Ser. I 335 (2002) 985–988. Our proof is based on an analytic description of the Hamiltonian flow on the fibers of the integral map of this system.

MSC: 34C15; 34C20; 37J35; 53D20; 55R55

Keywords: Liouville integrable system; Singular toric fibration; Monodromy

1. Introduction

Recently, Nekhoroshev et al. [9] introduced the concept of fractional (or generalized) monodromy using an explicit integrable perturbation of a 1:−2 resonant oscillator. Fractional monodromy is a substantial generalization of the usual (integer) monodromy of integrable Hamiltonian systems, discovered by Duistermaat [5] in the 1980s, as the simplest topological obstruction to the existence of global action-angle variables. In this paper we give an analytic proof of the existence of fractional monodromy for an integrable oscillator in 1:−2 resonance using a suit-

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able continuous basis of homology classes. In contrast to [9,10] our argument relies heavily on the dynamics of our system, because the homology classes we use are constructed from integral curves of certain vector fields related to the Hamiltonian vector fields of our integrable system. Our proof uncovers a subtle relation between the geometry of the singular fibration given by the level sets of the integrals of the resonant oscillator and the dynamics of its associated Hamiltonian vector fields. Because we regard this paper as a dynamical complement to [9,10], we refrain from discussing general geometric aspects, which the reader can find in [10]. For more introduction to the subject and its relevance to applications see [6].

1.1. Definition of integer monodromy

We summarize the standard approach to monodromy by Duistermaat and Cushman [4,5] and others. Consider a two degree of freedom Hamiltonian system on phase space $\mathbf{R}^4$ with coordinates $\xi = (x, p_x, y, p_y)$ and standard symplectic form $\omega = dx \wedge dp_x + dy \wedge dp_y$. Assume that the system is integrable, that is, there are two integrals F_1, F_2 whose Poisson bracket vanishes. We study the topology of the *integrable fibration* $\mathcal{F}$ whose leaves are defined by the fibers $F^{-1}(f) = F^{-1}(f_1, f_2) = F_1^{-1}(f_1) \cap F_2^{-1}(f_2)$ of the *integral map*

$$F : \mathbf{R}^4 \to \mathbf{R}^2 : \xi \mapsto \big(F_1(\xi), F_2(\xi)\big). \tag{1}$$

We assume that every fiber of F is connected and compact. Let $\mathcal{R} \subseteq \mathbf{R}^2$ be the image of F and $\mathcal{R}_{\mathrm{reg}} \subset \mathcal{R}$ be the set of regular values in the image of F. If $f \in \mathcal{R}_{\mathrm{reg}}$, we say that the fiber $F^{-1}(f)$ is regular. A regular fiber of F is a nonempty smooth 2-dimensional manifold, which is a 2-torus using the Liouville–Arnold theorem [1–3,8]. In addition, in a sufficiently small open neighborhood $U \subset \mathcal{R}_{\mathrm{reg}}$ of $f \in \mathcal{R}_{\mathrm{reg}}$ the map $F|F^{-1}(U) : F^{-1}(U) \to U$ is a trivial torus bundle, that is, $F^{-1}(U) = U \times \mathbf{T}^2$, see [3, Appendix D]. A question remains whether $F|F^{-1}(\mathcal{R}_{\mathrm{reg}}) : F^{-1}(\mathcal{R}_{\mathrm{reg}}) \to \mathcal{R}_{\mathrm{reg}}$ is *globally* a trivial 2-torus bundle.

Nontrivial topology in the integrable fibration $\mathcal{F}$ can arise from the presence of a singular fiber $F^{-1}(c)$, where the point $c \in \mathcal{R}$ is an *isolated critical value* of the integral map F. In other words the rank of $DF(\xi)$ is less than 2 for some $\xi \in F^{-1}(c)$ and there exists an open disk $D \subset \mathcal{R}$ such that $c \in D$ and $D \setminus \{c\} \subseteq \mathcal{R}_{\mathrm{reg}}$. To characterize the topology of $\mathcal{F}$ near c, we choose a closed path Γ in $D \setminus \{c\}$ as shown in Fig. 1. We determine the *monodromy* of the torus bundle $F|F^{-1}(\Gamma) : F^{-1}(\Gamma) \to \Gamma$ using standard methods from the theory of fiber bundles.

The classifying map χ of the bundle $F|F^{-1}(\Gamma) : F^{-1}(\Gamma) \to \Gamma$ induces a map

$$\chi_* : \mathrm{H}_1\big(\mathbf{T}^2_{f_0}, \mathbf{Z}\big) \to \mathrm{H}_1\big(\mathbf{T}^2_{f_0}, \mathbf{Z}\big)$$

on the first homology group $\mathrm{H}_1(\mathbf{T}^2_{f_0}, \mathbf{Z})$ of $\mathbf{T}^2_{f_0} = F^{-1}(f_0)$ where $f_0 \in \Gamma$. Therefore, instead of looking at the smooth bundle $F^{-1}(\Gamma) \to \Gamma$ of 2-tori over the loop Γ, we may look at the smooth bundle $\coprod_{f \in \Gamma} \mathrm{H}_1(\mathbf{T}^2_f, \mathbf{Z}) \to \Gamma$ of rank 2 lattices $\mathrm{H}_1(\mathbf{T}^2_f, \mathbf{Z})$ over Γ. The map χ_* is called the *monodromy map* of the bundle $F^{-1}(\Gamma) \to \Gamma$; it does not depend on the particular choice of Γ, but only on the homotopy class of $\Gamma \subset \mathcal{R}_{\mathrm{reg}}$ within the space of closed curves in $\mathcal{R}_{\mathrm{reg}}$. We obtain a homomorphism from the fundamental group $\pi_1(\mathcal{R}_{\mathrm{reg}})$ of $\mathcal{R}_{\mathrm{reg}}$ to the group of automorphisms of $\mathrm{H}_1(\mathbf{T}^2_{f_0}, \mathbf{Z})$, which is called the monodromy map on $\mathrm{H}_1(\mathbf{T}^2_{f_0}, \mathbf{Z})$.

Choosing a basis for $\mathrm{H}_1(\mathbf{T}^2_{f_0}, \mathbf{Z})$, the monodromy map is a 2×2 matrix $M \in \mathrm{SL}(2, \mathbf{Z})$, which is called the *monodromy matrix* along Γ. If M is not the identity, then the bundle $F^{-1}(\Gamma) \to \Gamma$

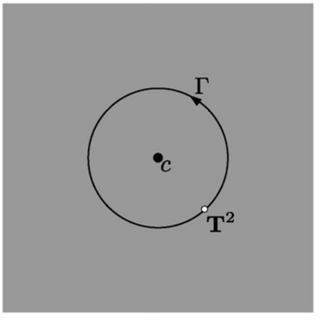

Fig. 1. In the case of standard monodromy the closed path Γ encircles an isolated critical value c of the integral map F. All points on Γ are regular values of F and therefore lift to regular tori $\mathbf{T}^2$.

of 2-tori over Γ is nontrivial and hence the whole foliation $\mathcal{F}$ is also nontrivial. In order to compute the monodromy matrix M of the torus bundle $F^{-1}(\Gamma) \to \Gamma$, we choose a basis of $\mathrm{H}_1(\mathbf{T}^2_{f_0}, \mathbf{Z})$ and then transport this basis *continuously* around Γ until we reach again f_0. In this way we obtain a *new* basis of $\mathrm{H}_1(\mathbf{T}^2_{f_0}, \mathbf{Z})$. The monodromy matrix M is the matrix that gives the transformation between these two bases.

Remark 1. This approach, using a continuation of a basis of the first homology group H_1, is equivalent to continuing a basis of the fundamental group π_1 of the 2-tori along Γ or a basis of the *period lattice PL*. The period lattice approach was introduced by Duistermaat [5] and is based on the fact that the period lattice bundle over Γ is isomorphic to the H_1 bundle over Γ.

1.2. Definition of fractional monodromy: the 1:−2 *oscillator*

In the case of generalized monodromy the bundle formed from fibers of the integral map F over Γ is not a $\mathbf{T}^2$ bundle, because for a finite number of points $c \in \Gamma$, $F^{-1}(c)$ is 'weakly' singular. In particular, this means that we do not have a smooth torus bundle over Γ. To characterize this singular bundle we pick a regular value $f_0 \in \Gamma$ of F, consider homology classes in a rank 2 *sublattice* $\mathcal{H}_1(\mathbf{T}^2_{f_0})$ of $\mathrm{H}_1(\mathbf{T}^2_{f_0}, \mathbf{Z})$, continue these classes along Γ, and obtain the monodromy map μ_Γ which is an automorphism of $\mathcal{H}_1(\mathbf{T}^2_{f_0})$. Note that $\mathcal{H}_1(\mathbf{T}^2_{f_0})$ and $\mathrm{H}_1(\mathbf{T}^2_{f_0}, \mathbf{Z})$ have the *same* rank. This means that the singularity of $F^{-1}(c)$ is weak enough for a sufficiently large subset of homology classes in $\mathrm{H}_1(\mathbf{T}^2_{f_0}, \mathbf{Z})$ to be continued. See [10] for a more rigorous and general discussion.

A concrete example of an integral map (1) which exhibits generalized monodromy was introduced in [9,10] following an intuitive idea of Zhilinskií[2] [11,12]. Consider the integral map

$$F_1(\xi) = J(\xi) = \frac{1}{2}\left(x^2 + p_x^2\right) - \left(y^2 + p_y^2\right), \tag{2a}$$

$$F_2(\xi) = H(\xi) = \sqrt{2}\left(\left(x^2 - p_x^2\right)p_y + 2xyp_x\right) + 2\epsilon\left(x^2 + p_x^2\right)\left(y^2 + p_y^2\right), \tag{2b}$$

[2] Private communication with (RC), University of Warwick, April 2002.

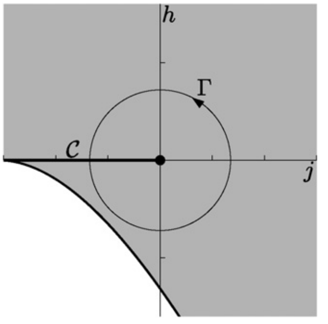

Fig. 2. The image of $\mathcal{EM}$ (2) near the singular value $(0,0)$. Regular values are shaded grey, critical values are shown by bold curves. The closed path Γ encircles $(0,0)$ and intersects the critical line segment $\mathcal{C}$ (3) of weak critical values once and there transversely.

which is a modification of the one used in [9]. The reason of this modification is explained in Appendix D. There we show that the integral map (2) and the original integral map in [9] define the same foliation of $\mathbf{R}^4$. See also Remark 2 below.

The function J (2a) is the Hamiltonian of a linear oscillator whose frequencies are in 1:−2 resonance. J generates a flow which is periodic on $\mathbf{R}^4 \setminus \{0\}$ with period 2π except on $\{x = p_x = 0\} \setminus \{0\}$ where the period is π. J is the *momentum* of the $\mathbf{S}^1$ Lie symmetry induced by this flow. The *energy* H (2b) is a constant of motion of this oscillator. The last term in (2b) ensures that the fibers of F are compact, the parameter $\epsilon > 0$ being a convenient scaling factor. We call the integral map F, the *energy-momentum* map $\mathcal{EM}(\xi) = (J(\xi), H(\xi))$. (Note that in the traditional notation used here, the value of momentum precedes that of energy so that momentum-energy map would be a more precise name.)

The set of critical values of $F = \mathcal{EM}$ near the singular value at the origin $(0,0)$ is shown in Fig. 2. In particular we are interested in the open line segment

$$\mathcal{C} = \left\{ c = (j, 0) \in \mathbf{R}^2 \colon 0 < -j < \frac{1}{2\epsilon^2} \right\} \tag{3}$$

of weakly singular values c. The weakly singular fibers $\mathcal{EM}^{-1}(c)$ are two-dimensional compact semi-algebraic sets whose singular set is the critical circle $\mathbf{S}^1_c = \mathcal{EM}^{-1}(c) \cap \{x = p_x = 0\}$.

Remark 2. It can be seen that H in (2) is defined so that $dh/dj = 0$ on $\mathcal{EM}^{-1}(c)$ for $c \in \mathcal{C}$. This fulfills the necessary condition for the orbits of the flow of X_H on $\mathcal{EM}^{-1}(c) \setminus \mathbf{S}^1_c$ to approach the singular circle $\mathbf{S}^1_c$ *transversely* (i.e., without spiraling infinitely along it) and the one sided limit of the rotation angle for the flow of X_H can be defined as f moves on Γ toward $c = \Gamma \cap \mathcal{C}$. Our work relies on this property of (2).

Following [10] we can show that for each $c \in \mathcal{C}$ the fiber $\mathcal{EM}^{-1}(c)$ is a *curled torus* consisting of the π-periodic orbit $\mathbf{S}^1_c$ together with its stable and unstable manifolds. The origin $(0,0)$ lifts to a different singular fiber called a *curled pinched torus* which is topologically equivalent to an once pinched 2-torus. In Appendix B we parameterize the set of critical values of $\mathcal{EM}$ and in Appendix C we describe the topology of the singular fibers of $\mathcal{EM}$ and explain how and why $\mathcal{EM}^{-1}(c)$ is 'curled.'

Consider now the closed path Γ discussed in the beginning of this section. Γ encircles $(0,0)$ and transversely intersects $\mathcal{C}$ once at $c \neq (0,0)$ but otherwise lies in the set of regular values $\mathcal{R}_{\mathrm{reg}}$ of $F = \mathcal{EM}$ (see Fig. 2). Since $\mathcal{EM}^{-1}(c)$ is not a regular 2-torus we cannot define the usual monodromy along Γ. Nevertheless, we can define generalized monodromy. The main idea of [9,10] is to find an index-2 sublattice $\mathcal{H}_1(\mathbf{T}^2_{f_0})$ of $\mathrm{H}_1(\mathbf{T}^2_{f_0}, \mathbf{Z})$, for which the monodromy map μ_Γ is defined. In particular we have

Theorem 1. (Cf. [9,10].) *In a properly chosen basis of* $\mathcal{H}_1(\mathbf{T}^2_{f_0})$ *the generalized monodromy map* μ_Γ *has matrix* $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$.

Remark 3. Extended formally to the whole $\mathrm{H}_1(\mathbf{T}^2_{f_0}, \mathbf{Z})$, μ_Γ has a matrix M with rational entries, that is, $M \in \mathrm{SL}(2, \mathbf{Q})$. Hence the name *fractional monodromy* is used as an alternative for *generalized monodromy*.

The authors of [9,10] formulate a similar theorem. Their proof, suggested in [9] and detailed in [10], is geometric. There the authors construct a cycle basis of the fundamental group π_1 of $\mathbf{T}^2_f$ for $f \in \Gamma \setminus \{c\}$, where $c \in \mathcal{C}$. One cycle is given by a closed integral curve of X_J on $\mathbf{T}^2_f$. The other is defined by the intersection of a fixed 3-dimensional Poincaré section σ to the flow of X_H on the h-level set of H. For a finite set of $f \in \Gamma \setminus \{c\}$, the intersection $\sigma \cap \mathbf{T}^2_f$ has singularities. To circumvent this difficulty, the authors use a homotopy.

In this work we give an analytic proof of Theorem 1 using appropriate periodic vector fields and constructing closed non-null-homotopic paths on the tori as integral curves of these vector fields. In this way we connect the dynamics of X_H and X_J to the geometry of the fibration. These closed paths are concrete cycles which represent elements of the sublattice $\mathcal{H}_1(\mathbf{T}^2_{f_0})$ of $\mathrm{H}_1(\mathbf{T}^2_{f_0}, \mathbf{Z})$. We refer to these closed paths as the *homology class representatives*.

In Section 2, we discuss the concept of the rotation angle and first return time used in the construction of our periodic vector fields. We study the properties of the rotation angle and the first return time analytically. We define a modified system for which the first return time is finite even on $F^{-1}(c)$, $c \in \mathcal{C}$. In Section 3, we consider the behavior of concrete representatives of the first homology group of $\mathbf{T}^2_f$ for $f \in \Gamma \setminus (\Gamma \cap \mathcal{C})$. We continue these classes along Γ and compute their variation when we have made a complete tour of Γ. This proves Theorem 1. Proofs of technical facts are given in the appendices.

2. Dynamical construction of cycle bases

2.1. Local action-angle variables

We recall the construction of local action-angle variables that is the cornerstone of both the period lattice approach in [3–5] and our homology group approach. There local actions are used to define cycle bases on regular tori $\mathbf{T}^2_f$. More details and complete proofs can be found in [3].

Suppose that $f = (j, h) \in \mathcal{R}_{\mathrm{reg}}$ is a regular value in the image of $\mathcal{EM}$. We look at the vector fields X_H and X_J on the invariant manifold $\mathcal{EM}^{-1}(f)$, which is a smooth 2-dimensional torus $\mathbf{T}^2_f$. The flow φ^t_J of X_J, given by

$$\varphi^t_J : \mathbf{S}^1 \times \mathbf{R}^4 \to \mathbf{R}^4 : (t, \xi) \mapsto R(t)\xi = \mathrm{diag}(R_t, R_{-2t})\xi, \tag{4}$$

where $\xi = (x, p_x, y, p_y)$ and $R_t = \left(\begin{smallmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{smallmatrix}\right)$, is periodic of period 2π except on $\{x = p_x = 0\} \setminus \{0\}$, where its period is π. The point $0 \in \mathbf{R}^4$ is an equilibrium of X_J. On the other hand, the flow

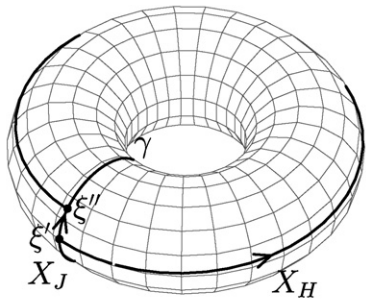

Fig. 3. The flow of the two vector fields X_H and X_J on the regular 2-torus $\mathbf{T}^2_{j,h}$; the flow of X_J is periodic while that of X_H is not.

φ_H of $X_H|\mathbf{T}^2_f$ is not periodic. Let γ be the periodic orbit on $\mathbf{T}^2_f$ traced out by an integral curve (4) of X_J starting at ξ' (see Fig. 3). Since X_H and X_J are linearly independent on $\mathbf{T}^2_f$, the integral curve $t \mapsto \varphi^t_H(\xi')$ of $X_H|\mathbf{T}^2_f$ starting at ξ' intersects γ transversely at ξ'' for the first positive time when $t = T(f)$. Because of transversality, locally $T(f)$ is smooth function of $f \in \mathcal{R}_{\text{reg}}$.

The twist of the flow of X_H on $\mathbf{T}^2_f$ is given by $\Theta(f) = \theta(\xi'') - \theta(\xi')$, where θ is an angle conjugate to the momentum J, see Section 2.2. Locally Θ is a smooth function of $f \in \mathcal{R}_{\text{reg}}$ which is defined up to addition of integer multiples of 2π. However, since $\mathcal{R}_{\text{reg}}$ is simply connected, Θ can be defined uniquely as a smooth function on the whole $\mathcal{R}_{\text{reg}}$.

Definition 1. The functions $\Theta(f)$ and $T(f)$ are called the rotation angle and the first return time of the vector field X_H with respect to X_J on $\mathbf{T}^2_f$, respectively.

Now define the vector fields

$$X_{I_1} : \mathbf{R}^4 \to \mathbf{R}^4 : \xi \mapsto X_{I_1}(\xi) = 2\pi X_J(\xi) \tag{5a}$$

and

$$X_{I_2} : \mathcal{EM}^{-1}(\mathcal{R}_{\text{reg}}) \to \mathbf{R}^4 : \xi \mapsto X_{I_2}(\xi) = -\Theta\big(\mathcal{EM}(\xi)\big) X_J(\xi) + T\big(\mathcal{EM}(\xi)\big) X_H(\xi) \tag{5b}$$

and consider their flows $\varphi_{X_{I_1}}$ and $\varphi_{X_{I_2}}$ on $\mathbf{T}^2_f$. Since $[X_H, X_J] = 0$, the flow of X_{I_2} on $\mathbf{T}^2_f$ is

$$\varphi^t_{X_{I_2}|\mathbf{T}^2_f} = \varphi^{T(f)t}_{X_H|\mathbf{T}^2_f} \circ \varphi^{-\Theta(f)t}_{X_J|\mathbf{T}^2_f}.$$

Lemma 1. *The flows* $\varphi^t_{X_{I_1}|\mathbf{T}^2_f}$ *and* $\varphi^t_{X_{I_2}|\mathbf{T}^2_f}$ *are periodic of* (*minimal*) *period* 1.

Proof. By construction (see Fig. 3),

$$\varphi^1_{X_{I_2}|\mathbf{T}^2_f}(\xi') = \varphi^{-\Theta(f)}_{X_J|\mathbf{T}^2_f} \circ \varphi^{T(f)}_{X_H|\mathbf{T}^2_f}(\xi') = \varphi^{-\Theta(f)}_{X_J|\mathbf{T}^2_f}(\xi'') = \xi'.$$

Since $\varphi_{X_H} \circ \varphi_{X_J} = \varphi_{X_J} \circ \varphi_{X_H}$, we obtain

$$\varphi^1_{X_{I_2}}\big(\varphi^s_{X_J}(\xi')\big) = \varphi^s_{X_J}(\xi')$$

for every $s \in [0, 1]$. Therefore $\varphi^1_{X_{I_2}}$ is the identity map on γ, which is a Poincaré cross section for the flow of X_H on $\mathbf{T}^2_f$. Since γ is the image of an arbitrary integral curve of X_J on $\mathbf{T}^2_f$, it follows that the flow of $X_{I_2}|\mathbf{T}^2_f$ is periodic of period 1. □

Remark 4. According to Lemma 1 the vector fields X_{I_i}, $i = 1, 2$, are the Hamiltonian vector fields of local actions I_i, $i = 1, 2$.

2.2. Computation of the rotation angle and the first return time

We calculate the time $T(f)$ of first return and the rotation angle $\Theta(f)$ of the flow of X_H on the regular 2-torus $\mathbf{T}^2_f$ leaf of the integrable fibration defined by $\mathcal{EM}$ (2). These computations are greatly simplified if we first reduce the $\mathbf{S}^1$ symmetry of our system which is generated by the flow φ^t_J of X_J. We use here some basic facts about reduction of our particular $\mathbf{S}^1$ action on $\mathbf{R}^4$. This $\mathbf{S}^1$ action has four basic quadratic polynomial invariants J and (π_1, π_2, π_3). Their explicit definition along with further details is given in Appendix A.

For j in the image of the momentum map J, the reduced phase space P_j is the semi-algebraic variety defined by

$$\pi_2^2 + \pi_3^2 = (\pi_1 + j)^2(\pi_1 - j) \quad \text{and} \quad \pi_1 \geqslant |j|. \tag{6}$$

The reduced Hamiltonian on P_j is

$$H_j = \pi_3 + \epsilon\big(\pi_1^2 - j^2\big). \tag{7}$$

The reduced dynamics is described by $\dot{\pi}_k = \{\pi_k, H_j\}$ for $k = 1, 2, 3$. In particular,

$$\dot{\pi}_1 = 4\pi_2. \tag{8}$$

When $f = (j, h)$ is a regular value of $\mathcal{EM}$, the orbit of X_{H_j} on P_j coincides with $H_j^{-1}(h) \cap P_j$, which is diffeomorphic to a circle, see Appendix C. We denote by π_1^- and π_1^+ the minimum and maximum values, respectively, attained by π_1 along such an orbit. A motion of the reduced vector field traces out the circle $H_j^{-1}(h) \cap P_j$ and thus is periodic with period $T(j, h)$.

To find the first return time and the rotation angle we observe that

Lemma 2. *The period of the reduced motion of X_{H_j} on P_j for energy $H_j = h$ is equal to the first return time of X_H on $\mathbf{T}^2_{j,h}$.*

Proof. The first return time on $\mathbf{T}^2_{j,h}$ is the time after which the trajectory of X_H returns to the same $\mathbf{S}^1$ orbit of φ_{X_J}. The lemma follows because different points in the reduced space lift to different φ_{X_J} orbits. □

We now compute the period $T(j, h)$. We have

$$T(j, h) = \int_0^{T(j,h)} \mathrm{d}t = 2\int_{\pi_1^-}^{\pi_1^+} \frac{\mathrm{d}\pi_1}{\dot{\pi}_1} = \frac{1}{2}\int_{\pi_1^-}^{\pi_1^+} \frac{\mathrm{d}\pi_1}{\pi_2}, \tag{9}$$

where the last equality follows from (8). From (6) and (7) we have

$$\pi_2^2 = S_{j,h}(\pi_1) = (\pi_1 - j)(\pi_1 + j)^2 - \bigl(h - \epsilon\bigl(\pi_1^2 - j^2\bigr)\bigr)^2. \tag{10}$$

Hence (9) becomes

$$T(j,h) = \frac{1}{2}\int_{\pi_1^-}^{\pi_1^+} \frac{\mathrm{d}\pi_1}{\sqrt{S_{j,h}(\pi_1)}}. \tag{11}$$

Note that $\pi_1^{\pm}$ are real roots of $S_{j,h}$ (10) that are greater than or equal to $|j|$. As (j,h) approaches a point $c = (j_*, 0)$ on the critical line segment $\mathcal{C}$ (3), the period $T(j,h)$ goes to infinity, because $\mathcal{EM}^{-1}(c)$ is the union of the stable and unstable manifolds of a hyperbolic periodic orbit of the smooth vector field X_H. Moreover, for (j,h) near $\mathcal{C}$, the polynomial $S_{j,h}$ has four real roots, two of which coincide on $\mathcal{C}$. This causes the integral (11) to blow up. A proof of this fact is given in Appendix E.3.

Next we determine the rotation angle of the flow of X_H on $\mathbf{T}^2_{j,h}$. Let $\theta = \tan^{-1}(x/p_x)$. The (multivalued) function θ is canonically conjugate to J, because $\{\theta, J\} = \mathcal{L}_{X_J}\theta = 1$. The time derivative of θ along an integral curve of X_H is single valued and is given by

$$\dot\theta = \mathcal{L}_{X_H}\theta = \frac{x\dot p_x - p_x\dot x}{x^2 + p_x^2}. \tag{12}$$

Explicit expressions for $\dot x$ and $\dot p_x$ come from Hamilton's equations for the integral curves of X_H. The function $\dot\theta$ is ϕ^J-invariant, since

$$\{J, \dot\theta\} = \bigl\{J, \{\theta, H\}\bigr\} = \bigl\{\{J,\theta\}, H\bigr\} + \bigl\{\theta, \{J, H\}\bigr\} = \{-1, H\} = 0.$$

Therefore we can express $\dot\theta$ in terms of the invariants J, π_1, π_2 and π_3. A short computation gives

$$\dot\theta = 2\,\frac{h}{j + \pi_1}. \tag{13}$$

Then the *rotation angle* of the flow of X_H on $\mathbf{T}^2_{j,h} = \mathcal{EM}^{-1}(j,h)$ is

$$\begin{aligned}\Theta(j,h) &= \int_0^{\Theta(j,h)} \mathrm{d}\theta = \int_0^{T(j,h)} \dot\theta\,\mathrm{d}t = 4h\int_{\pi_1^-}^{\pi_1^+} \frac{1}{j+\pi_1}\frac{\mathrm{d}\pi_1}{\dot\pi_1} \\ &= h\int_{\pi_1^-}^{\pi_1^+} \frac{1}{j+\pi_1}\frac{\mathrm{d}\pi_1}{\sqrt{S_{j,h}(\pi_1)}}.\end{aligned} \tag{14}$$

Contrary to $T(j,h)$ and because of the specific choice of H (2b), the rotation angle $\Theta(j,h)$ (14) has a finite limit as $h \to 0$, see Appendix D. More precisely, we assert

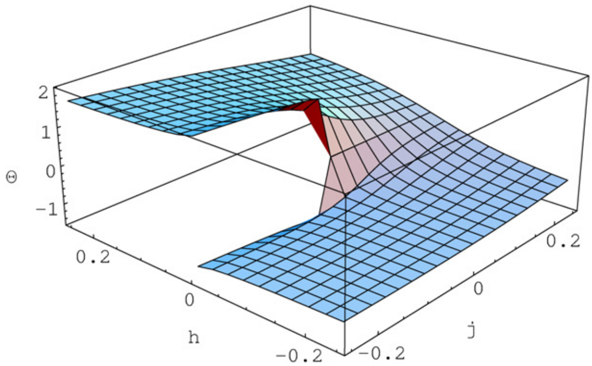

Fig. 4. Rotation number $\Theta(j,h)$ near (0, 0). The discontinuity of $\Theta(j,h)$ begins at (0, 0) and continues along the critical line segment $\mathcal{C}$.

Lemma 3. *For* $j \in (-\frac{1}{2}\epsilon^{-2}, 0)$ *we have*

$$\lim_{h\to 0^+} \Theta(j,h) = \frac{\pi}{2} + \sin^{-1}\bigl(\epsilon\bigl(2|j|\bigr)^{1/2}\bigr) \tag{15a}$$

and

$$\lim_{h\to 0^-} \Theta(j,h) = -\frac{\pi}{2} + \sin^{-1}\bigl(\epsilon\bigl(2|j|\bigr)^{1/2}\bigr). \tag{15b}$$

The proof is given in Appendix E.5. In particular, this means that there is a jump of π of the rotation number at $\mathcal{C}$, see Fig. 4.

2.3. Time rescaled dynamics

In Section 2.1 we defined the vector field X_{I_2} which has periodic flow on $\mathbf{T}^2_f$, $f \in \mathcal{R}_{\text{reg}}$. This vector field cannot be defined on the curled tori because $\lim_{h\to 0} T(j,h) = \infty$ for $j < 0$, see Appendix E.3. However, on $\mathbf{R}^4 \setminus \{x = p_x = 0\}$ we can define the vector field

$$X(\xi) = \frac{1}{p_x^2 + x^2} X_H(\xi), \tag{16}$$

which is a time rescaling of X_H.

Lemma 4. *X is a smooth incomplete vector field on $\mathbf{R}^4 \setminus \{x = p_x = 0\}$, which commutes with X_J and leaves the set $\mathcal{EM}^{-1}(f) \setminus \{x = p_x = 0\}$, $f \in \mathcal{R}$ invariant.*

Proof. The vector field X is not complete because an integral curve starting at a point in $\mathcal{EM}^{-1}(j,0)$ with $j < 0$ reaches a point in $\{x = p_x = 0\}$ in finite time, see Appendix F. Since $g : \mathbf{R}^4 \setminus \{x = p_x = 0\} \to \mathbf{R} : (x, p_x, y, p_y) \to (p_x^2 + x^2)^{-1}$ is invariant under the flow of X_J, the vector fields X and X_J commute where they are both defined. The last assertion in the lemma follows because for any smooth function $G : \mathbf{R}^4 \to \mathbf{R}$ we have

$$\mathcal{L}_X G = \mathcal{L}_{gX_H} G = g(\mathcal{L}_{X_H} G) = g\{G, H\}.$$

In particular, $\mathcal{L}_X H = \mathcal{L}_X J = 0$. $\square$

We need to define the first return time $\tau(j,h)$ and the rotation angle $\widehat{\Theta}(j,h)$ of the vector field X with respect to X_J. For $(j,h) \in \mathcal{R}_{\text{reg}}$ this is straightforward.

Consider now a weak singular value $c \in \mathcal{C}$ in (3), that is, $c = (j,0)$ with $-\frac{1}{2}\epsilon^{-2} < j < 0$ and $\xi \in \mathcal{EM}^{-1}(c) \setminus \{x = p_x = 0\}$ a regular point of the curled torus. If $\tau_+(\xi)$ and $\tau_-(\xi)$ are the positive and negative times needed by the integral curve of X starting at ξ to reach $\{x = p_x = 0\}$, then we can define the first return time $\tau(\xi)$ of X with respect to X_J to be $\tau(\xi) = \tau_+(\xi) + |\tau_-(\xi)|$. Since $\tau(\xi)$ is the same for all regular points $\xi \in \mathcal{EM}^{-1}(c)$, we may define $\tau(j,0) = \tau(\xi)$, where $(j,0) = \mathcal{EM}(\xi)$. In Appendix F we show that $\lim_{h\to 0} \tau(j,h) = \tau(j,0)$ for any $(j,0) \in \mathcal{C}$. Moreover, in Appendix E.4 we prove

Lemma 5. *The first time $\tau : \mathcal{R} \to \mathbf{R}$ of return of the vector field X (16) with respect to X_J is a smooth function on $\mathcal{R}_{\text{reg}}$ and is continuous on $\mathcal{C}$.*

We note that the *rotation angle* $\widehat{\Theta}(j,h)$ of the vector field X with respect to X_J equals the rotation angle $\Theta(j,h)$ (14) of X_H, because

$$\widehat{\Theta}(j,h) = \int_0^{\tau(j,h)} \frac{\mathrm{d}\theta}{\mathrm{d}s}\mathrm{d}s = \int_0^{T(j,h)} \frac{\mathrm{d}\theta}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t}\,\mathrm{d}t = \int_0^{T(j,h)} \frac{\mathrm{d}\theta}{\mathrm{d}t}\,\mathrm{d}t = \Theta(j,h).$$

Hence below we will drop the hat over Θ.

Similar to X_{I_1} and X_{I_2} in Eq. (5b) we define the modified (or time-rescaled) vector fields

$$X_1(\xi) = 2\pi X_J(\xi) \tag{17}$$

and

$$X_2(\xi) = -\Theta\big(\mathcal{EM}(\xi)\big)X_J(\xi) + \tau\big(\mathcal{EM}(\xi)\big)X(\xi), \tag{18}$$

whose basic properties are given by

Lemma 6. *The modified vector fields X_1 (17) and X_2 (18) are smooth. Their restrictions to any torus $\mathbf{T}^2_f$ with $f \in \mathcal{R}_{\text{reg}}$ are linearly independent, have periodic flows of period 1, and commute.*

2.4. Choice of homology basis on regular fibers

Recall that the flow of X_J is periodic with period 2π on $\mathbf{R}^4 \setminus \{x = p_x = 0\}$. Let $[\beta_1]$ be the homology class of the closed integral curve

$$\beta_1 : [0,1] \to \mathbf{T}^2_f : t \mapsto \varphi^t_{X_1}(\xi)$$

of the vector field X_1 starting at $\xi \in \mathbf{T}^2_f$. Note that $[\beta_1]$ does not depend on ξ. A second independent homology class $[\beta_2]$ on $\mathbf{T}^2_f$ can be constructed from the closed integral curve

$$\beta_2 : [0,1] \to \mathbf{T}^2_f : t \mapsto \varphi^t_{X_2}(\xi)$$

of the vector field X_2 starting at ξ. Again $[\beta_2]$ does not depend on the choice of ξ. We use $\{[\beta_1],[\beta_2]\}$ as basis for $\mathrm{H}_1(\mathbf{T}^2_f, \mathbf{Z})$, when $f \in \mathcal{R}_{\text{reg}}$.

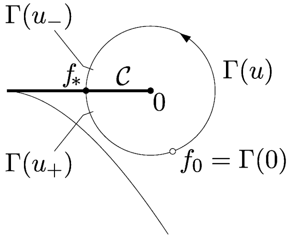

Fig. 5. The closed path Γ in the base of the 1:−2 fibration.

3. Proof of fractional monodromy

In this section we prove Theorem 1. Consider the smooth closed path

$$\Gamma : [0,1] \to \mathcal{R} : u \mapsto \Gamma(u) = \big(j(u), h(u)\big), \tag{19}$$

and $f_0 = \Gamma(0) = \Gamma(1) \notin \mathcal{C}$. The curve Γ is positively oriented and encircles the origin $0 = (0,0)$ once. It intersects the critical line segment $\mathcal{C}$ transversely once at $f_* = \Gamma(u_*)$, where $u_* \in (0,1)$, but otherwise lies in $\mathcal{R}_{\mathrm{reg}}$, see Fig. 5.

Let $u_\pm$ be close to u_* so that $[u_-, u_+] \subseteq [0,1]$. Note that Lemmas 10–15, given later in this section, are formulated for every $u \in [0,1]$. We will focus here on the neighborhood of $u_* \in [u_-, u_+]$ and then later consider necessary extensions to $[0,1]$. We first determine the behavior in $\mathcal{EM}^{-1}(\Gamma)$ of certain oriented closed curves $\gamma_i(u)$ with $i = 1, 2$, which lie in $\Lambda_u = \mathcal{EM}^{-1}(\Gamma(u))$ and which depend *continuously* on u. Their homology classes $[\gamma_i(u)]$ will be independent and therefore will form a basis of a sublattice of $\mathrm{H}_1(\Lambda_u, \mathbf{Z})$ for $u \in [u_-, u_+] \setminus \{u_*\}$.

Since J is a globally defined action,

$$\gamma_1(u) = \beta_1 : [0,1] \to \Lambda_u : t \mapsto \varphi^t_{X_1}(\xi) \tag{20}$$

is a natural choice of one of the curves. Using the properties of the integral curves of the vector field X_1 (17), in Lemma 9 we show that the homology class $[\gamma_1(u)]$ of $\gamma_1(u)$ does not depend on either ξ or $u \in [u_-, u_+]$.

The construction of $\gamma_2(u)$ is at the heart of our proof of Theorem 1. It is more involved than that of $\gamma_1(u)$. Because the rotation angle Θ of the vector field X (16) has a jump of $-\pi$ as u increases through u_*, see Lemma 3 and Fig. 4, the function

$$\vartheta : [u_-, u_+] \to \mathbf{R} : u \mapsto \begin{cases} \Theta(\Gamma(u)) - \pi, & u \in [u_-, u_*), \\ \lim_{u \to u_*^+} \Theta(\Gamma(u)), & u = u_*, \\ \Theta(\Gamma(u)), & u \in (u_*, u_+] \end{cases} \tag{21}$$

is continuous at u_*. The vector field

$$Z^u(\xi) = -\vartheta(u) X_J(\xi) + \tau\big(\Gamma(u)\big) X(\xi) \tag{22}$$

on Λ_u is well defined for all $u \in [u_-, u_+]$, because $X(\xi) = X_J(\xi)$ when ξ is a critical point of $\mathcal{EM}$, and depends continuously on u, because ϑ and τ are continuous, see Section 2.3 and Lemma 5. Now consider the family of initial points

$$\xi_\pm(u) = \big(x_\pm(u), p_x(u), y(u), p_y(u)\big) \in \Lambda_u, \tag{23a}$$

where

$$x_\pm(u) = \pm\sqrt{\pi_1^+(u) + j(u)}, \qquad p_x(u) = 0, \tag{23b}$$

$$y(u) = 0, \qquad p_y(u) = \frac{h(u) - \epsilon(\pi_1^+(u)^2 - j(u)^2)}{\sqrt{2}(\pi_1^+(u) + j(u))}, \tag{23c}$$

and $\pi_1^+(u) = \max_{H^{-1}_{j(u)}(h(u))} \pi_1$, see Section 2.2. The properties of $\xi_\pm(u)$ are detailed in Lemma 11. We will use $\xi_\pm(u)$ to construct $\gamma_2(u)$ as the union of the integral curves

$$\gamma_2^a(u) : \left[-\frac{1}{2}, \frac{1}{2}\right] \to \Lambda_u : t \mapsto \varphi^t_{Z^u}\big(\xi_+(u)\big) \tag{24a}$$

and

$$\gamma_2^b(u) : \left[-\frac{1}{2}, \frac{1}{2}\right] \to \Lambda_u : t \mapsto \varphi^t_{Z^u}\big(\xi_-(u)\big). \tag{24b}$$

Lemmas 12 and 13 describe the geometric behavior of the family $u \mapsto \gamma_2(u)$ illustrated in Fig. 6. Specifically, in Lemma 12 we show that for $u \in [u_-, u_*]$ we have

$$\gamma_2^a(u)\left(\frac{1}{2}\right) = \gamma_2^b(u)\left(-\frac{1}{2}\right) = \chi_+(u) \quad \text{and} \quad \gamma_2^a(u)\left(-\frac{1}{2}\right) = \gamma_2^b(u)\left(\frac{1}{2}\right) = \chi_-(u);$$

while for $u \in (u_*, u_+]$ we have

$$\gamma_2^a(u)\left(-\frac{1}{2}\right) = \gamma_2^a(u)\left(\frac{1}{2}\right) = \overline{\chi}_+(u) \quad \text{and} \quad \gamma_2^b(u)\left(-\frac{1}{2}\right) = \gamma_2^b(u)\left(\frac{1}{2}\right) = \overline{\chi}_-(u).$$

Moreover, $\chi_+(u_*) = \chi_-(u_*) = \chi_*$, while $\chi_-(u) \neq \chi_+(u)$, when $u \in [u_-, u_*)$. In Lemma 13 we show that the curve $\gamma_2(u)$ is continuous for all $u \in [u_-, u_+]$. Furthermore, in Lemma 14 we verify that the family $u \mapsto \gamma_2(u)$ is continuous on [0, 1] and therefore on $[u_-, u_+]$.

Once the appropriate family $u \mapsto (\gamma_1(u), \gamma_2(u))$ of the oriented continuous closed curves in Λ_u has been constructed, we express the homology classes $[\gamma_1(u)]$ and $[\gamma_2(u)]$ for $u \neq u_*$ in terms of the basis $\{[\beta_1(u)], [\beta_2(u)]\}$ of $\mathrm{H}_1(\mathbf{T}^2_{(j,h)}, \mathbf{Z})$ chosen in Section 2.4. Clearly, $[\gamma_1(u)] = [\beta_1(u)]$. In Lemma 15 a computation shows that

$$\big[\gamma_2(u)\big] = \begin{cases} [\beta_1(u)] + 2[\beta_2(u)], & u \in [u_-, u_*), \\ 2[\beta_2(u)], & u \in (u_*, u_+]. \end{cases} \tag{25}$$

(Of course, for $u \in (u_*, u_+]$ this follows directly from the construction of $\gamma_2(u)$ illustrated in Fig. 6, bottom.) From Lemma 15 we conclude that for every regular $u \in [u_-, u_+] \setminus \{u_*\}$, the

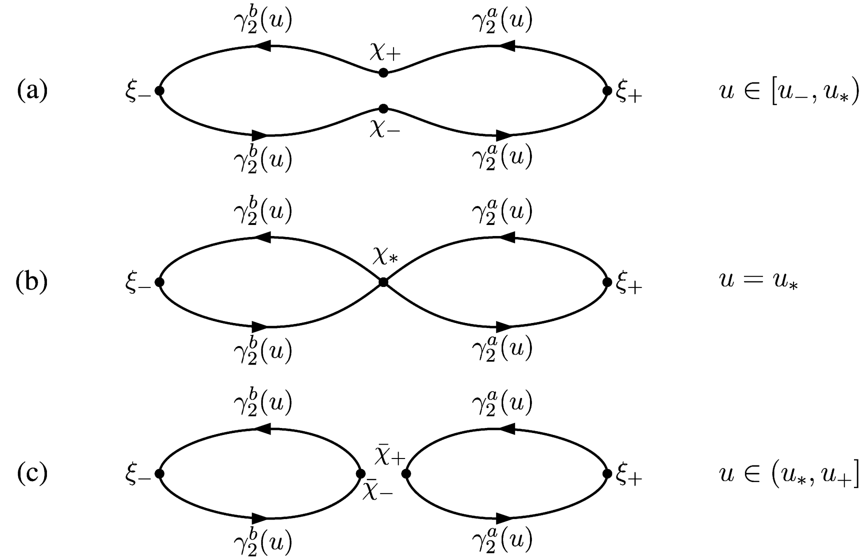

Fig. 6. Schematic representation of the paths $\gamma_2^a(u)$ and $\gamma_2^b(u)$ on the fiber $\Lambda_u = \mathcal{EM}^{-1}(\Gamma(u))$ which join together when $u = u_*$ (middle) so that the deformation of the composite path $\gamma_2(u) = \gamma_2^a(u) \cup \gamma_2^b(u)$ is continuous at u_*.

homology classes $[\gamma_1(u)]$ and $[\gamma_2(u)]$ generate the index-2 subgroup $\mathcal{H}_1(\mathbf{T}^2_{\Gamma(u)})$ of $\mathrm{H}_1(\mathbf{T}^2_{\Gamma(u)})$. Now we can prove

Lemma 7. *For $u \in [u_-, u_+]$ the only homology classes in $\mathrm{H}_1(\mathbf{T}^2_{\Gamma(u)}, \mathbf{Z})$ that vary continuously as u crosses u_* are those which lie in the rank 2 sublattice $\mathcal{H}_1(\mathbf{T}^2_{\Gamma(u)}, \mathbf{Z})$ of $\mathrm{H}_1(\mathbf{T}^2_{\Gamma(u)}, \mathbf{Z})$ spanned by $[\gamma_1(u)]$ and $[\gamma_2(u)]$ with $u \neq u_*$.*

Proof. We have already shown that $[\gamma_1(u)]$ and $[\gamma_2(u)]$ are linearly independent and continuous, and that they span $\mathcal{H}_1(\mathbf{T}^2_{\Gamma(u)}, \mathbf{Z})$. From (25) we can see that all homology classes of $\mathrm{H}_1(\mathbf{T}^2_{\Gamma(u)}, \mathbf{Z})$ not contained $\mathcal{H}_1(\mathbf{T}^2_{\Gamma(u)}, \mathbf{Z})$ lie in the sublattice spanned by $\{[\beta_2(u)]\}$. Reading Fig. 6 from bottom to top, we see that the curve $\gamma_2^a(u)$, which is homologous to the closed loop $\beta_2(u)$ for $u \in (u_*, u_+]$, does *not* vary continuously as u decreases through u_*, because for $u \in [u_-, u_*)$ the curve $\gamma_2^a(u)$ is a segment with distinct end points $\chi_\pm$ and thus is not a cycle because it is not a closed loop. Therefore, the homology class $[\beta_2(u)]$ (and thus any class not in $\mathcal{H}_1$) cannot be continued across u_*. □

Let Γ be the positively oriented closed curve described at the beginning of this section. In order to compute the generalized monodromy map $\mu_\Gamma : \mathcal{H}_1(\mathbf{T}^2_{f_0}) \to \mathcal{H}_1(\mathbf{T}^2_{f_0})$ along Γ starting at f_0, we need to extend our construction of $[\gamma_2(u)]$ from $[u_-, u_+]$ to $[0, 1]$. We make some observations. First, the function ϑ (21) extends to the continuous function

$$\vartheta : [0,1] \to \mathbf{R} : u \mapsto \begin{cases} \Theta(\Gamma(u)) - \pi, & u \in [0, u_*), \\ \lim_{u \to u_*^+} \Theta(\Gamma(u)), & u = u_*, \\ \Theta(\Gamma(u)), & u \in (u_*, 1]. \end{cases} \tag{21*}$$

Note that $\vartheta(0) = \vartheta(1) + \pi$. Second, since the vector field Z^u (22) is defined on $\mathcal{EM}^{-1}(\Gamma(u))$ and is smooth for each $u \in [0,1]$, the family $u \mapsto Z^u$ is continuous on $[0,1]$. Note that $Z^1 \neq Z^0$ are two *different* vector fields on the torus $\mathbf{T}^2_{f_0}$. Second, the curve $\gamma_2(u)$, defined before Fig. 6, is continuous for all $u \in [0,1]$. Of course, since X_1 does not depend on u, the curve $\gamma_1(u)$ is also continuous for all $u \in [0,1]$. In addition, these families have the properties given in Lemmas 12 and 13. Fourth, in Lemma 14 we show that the families $u \mapsto \gamma_1(u)$ and $u \mapsto \gamma_2(u)$ are continuous on $[0,1]$. For $\gamma_2(u)$ we use the continuity of the mapping $u \mapsto \xi_\pm(u)$ (23) on $[0,1]$. Note that $\xi_\pm(0) = \xi_\pm(1)$.

Remark 5. Even though Z^u, ϑ, and $u \mapsto \gamma_2(u)$ are continuous in u on the whole closed interval $[0,1]$, and in particular at points $u = 0$ and $u = 1$, considered as being defined on $\Gamma([0,1])$ they are discontinuous at f_0. In particular, ϑ has a jump of $-\pi$ at $\Gamma(1)$, since $\vartheta(1) = \vartheta(0) - \pi$. To circumvent possible confusion, we can use a semi-open interval $[0,1)$ and take the limit as $u \to 1^-$ in order to get statements about the point $u = 1$. To do this we use

$$\lim_{u\to 1^-} Z^u(\xi) = -\big(\Theta\big(\Gamma(0)\big) - \pi\big)X_J(\xi) + \tau\big(\Gamma(0)\big)X(\xi)$$

which is a smooth vector field on $\mathbf{T}^2_{f_0}$. Only then can we conclude that $\gamma_2(1) = \lim_{u\to 1^-} \gamma_2(u)$ exists and is a continuous closed curve on $\mathbf{T}^2_{f_0}$.

Now we are in a position to compute the monodromy matrix which corresponds to the generalized monodromy map μ_Γ. We have already seen at the beginning of this section that $[\gamma_1(u)]$ does not depend on the parameter u for $u \in [u_-, u_+]$. By the same argument, this holds for all $u \in [0,1]$. Therefore $[\gamma_1(1)] = [\gamma_1(0)]$.

Using Lemma 15 we see that $\mathcal{H}_1(\Lambda_u)$ is an index-2 subgroup of $\mathrm{H}_1(\Lambda_u, \mathbf{Z})$ for all $u \in [0,1] \setminus \{u_*\}$.

To compare $\mathcal{H}_1(\mathcal{EM}^{-1}(\Gamma(0)))$ with $\mathcal{H}_1(\mathcal{EM}^{-1}(\Gamma(1)))$ note that the cycle basis $\{[\beta_1],[\beta_2]\}$ on $\mathbf{T}^2_{f_0} = \mathcal{EM}^{-1}(\Gamma(0)) = \mathcal{EM}^{-1}(\Gamma(1))$, chosen as in Section 2.4, is obviously independent on the parameter u. In other words

$$\big\{\big[\beta_1(0)\big],\big[\beta_2(0)\big]\big\} = \big\{\big[\beta_1(1)\big],\big[\beta_2(1)\big]\big\} = \big\{[\beta_1],[\beta_2]\big\}_{f_0}.$$

So using Lemma 15 we obtain $[\gamma_2(0)] = [\beta_1(0)] + 2[\beta_2(0)]$ and $[\gamma_2(1)] = 2[\beta_2(1)] = 2[\beta_2(0)]$. Therefore, with respect to the basis $\{[\beta_1(0)], 2[\beta_2(0)]\}$ of $\mathcal{H}_1(\mathbf{T}^2_{\Gamma(0)})$ the generalized monodromy mapping μ_Γ is the linear mapping that sends $[\gamma_1(0)] = (1,0)^T$ and $[\gamma_2(0)] = (1,1)^T$ into $[\gamma_1(1)] = (1,0)^T$ and $[\gamma_2(1)] = (0,1)^T$, respectively. Thus we have proved

Lemma 8. *The matrix of the generalized monodromy map* $\mu_\Gamma : \mathcal{H}_1(\mathbf{T}^2_{\Gamma(0)}) \to \mathcal{H}_1(\mathbf{T}^2_{\Gamma(0)})$ *with respect to the basis* $\{[\beta_1(0)], 2[\beta_2(0)]\}$ *is* $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.[3]

We now prove the lemmas referred to in the preceding argument. We begin with

Lemma 9. *For any* $u \in [0,1]$ *and* $\xi \in \Lambda_u = \mathcal{EM}^{-1}(\Gamma(u))$, *the curve* $\gamma_1(u)$ *defined in* (20) *as an integral curve of the vector field* X_1 *starting in* ξ, *is a smooth closed curve on* Λ_u. *The mapping*

[3] Note that our convention for defining the monodromy matrix differs from that in [9,10] by transposition.

$u \mapsto \gamma_1(u)$ is continuous on $[0,1]$. The homology class $[\gamma_1(u)]$ of $\gamma_1(u)$ does not depend on either ξ or u.

Proof. Because J is a globally defined action (momentum), all integral curves $\gamma_1(u) = \beta_1$ of the vector field X_1 in (17) on all fibers Λ_u are closed smooth curves. For all regular tori, and in particular for Λ_u with $u \neq u_*$ see Lemma 6; for the curled torus Λ_{u_*} this can be verified by a computation.

When $\xi \notin \{x = p_x = 0\}$, integral curves of X_1 have period 1. Otherwise, the integral curve of X_1 lies on the critical circle $\mathbf{S}^1_{\Gamma(u_*)} = \Lambda_{u_*} \cap \{x = p_x = 0\}$ and closes after period $1/2$. However, the curve $\gamma_1(u_*)$ with $\xi \in \{x = p_x = 0\}$ traces out $\mathbf{S}^1_{\Gamma(u_*)}$ *twice* and is homologous to any integral curve of X_1 with starting point ξ on $\Lambda_{u_*} \setminus \mathbf{S}^1_{\Gamma(u_*)}$. This assures that $\gamma_1(u)$ is both continuous at u_* and independent of ξ.

By choosing a continuous family $u \mapsto \xi(u) \in \Lambda_u$, which is always possible, we obtain a continuous family $u \mapsto \gamma_1(u)$. Since the homology class $[\gamma_1(u)]$ does not depend on ξ, any such family results in the same continuous family $[\gamma_1(u)]$ of homology classes. Moreover, since the vector field X_1 does not depend on the parameter u, the curve $\gamma_1(u)$ is homologous to any period-1 integral curve of X_1 on Λ_u and therefore, $[\gamma_1(u)]$ does not change with u. □

Lemma 10. *The flow of Z^u is periodic for all $u \in [0, u_*) \cup (u_*, 1]$. For $u \in [0, u_*)$ its minimal period is 2; while for $u \in (u_*, 1]$ its minimal period is 1.*

Proof. It is easy to see from the definition of ϑ (21) that

$$Z^u|\mathbf{T}^2_{\Gamma(u)} = \begin{cases} \pi X_J|\mathbf{T}^2_{\Gamma(u)} + X_2|\mathbf{T}^2_{\Gamma(u)}, & u \in [0, u_*), \\ X_2|\mathbf{T}^2_{\Gamma(u)}, & u \in (u_*, 1]. \end{cases} \tag{26}$$

This means that with $\xi \in \mathbf{T}^2_{\Gamma(u)}$ we have

$$\varphi^t_{Z^u}(\xi) = \varphi^{\pi t}_{X_J} \circ \varphi^t_{X_2}(\xi), \tag{27a}$$

for $u \in [0, u_*)$; while for $u \in (u_*, 1]$ we have

$$\varphi^t_{Z^u}(\xi) = \varphi^t_{X_2}(\xi). \tag{27b}$$

When $u \in (u_*, 1]$, then $Z^u(\xi) = X_2(\xi)$. By construction of the modified first return time and the modified rotation angle with respect to X_J of the vector field, we see that the flow of Z^u on $\mathbf{T}^2_{\Gamma(u)}$ has minimal period 1. (In the spirit of Remark 5, consider $u \in (u_*, 1)$ and show that the lemma holds for $u = 1$ by taking the limit as $u \to 1^-$.)

When $u \in [0, u_*)$, then $Z^u = \pi X_J + X_2$ on $\mathbf{T}^2_{\Gamma(u)}$. Recall that $\varphi^1_{X_2}(\xi) = \xi$ for all $\xi \in \mathbf{T}^2_{\Gamma(u)}$. Moreover the integral curve of X_2 beginning at ξ intersects the integral curve of X_J beginning at ξ at times $t = n \in \mathbf{Z}$. Since X_J moves points only along the integral curve of X_J going through ξ we can have $\varphi^{\pi t}_{X_J} \circ \varphi^t_{X_2}(\xi) = \xi$ only when $t = n \in \mathbf{Z}$. Consequently, $\varphi^{\pi n}_{X_J}(\xi) = \xi$ only when n is an even integer. Therefore for $u \in [0, u_*)$ the minimal period of $Z^u|\mathbf{T}^2_{\Gamma(u)}$ is 2. □

Remark 6. In general, the flow of Z^u is not periodic outside of $\mathbf{T}^2_{\Gamma(u)}$, where $u \in [0, u_*) \cup (u_*, 1]$. Therefore Z^u can only serve to define a cycle representative on $\mathbf{T}^2_{\Gamma(u)}$.

Next we show

Lemma 11. *For all $u \in [0,1]$, the family of initial points $\xi_\pm$ defined in* (23) *is a smooth closed curve in* $\mathbf{R}^4$ *and* $\xi_\pm(u) \in \mathcal{EM}^{-1}(\Gamma(u))$ *for all* $u \in [0,1]$.

Recall from Section 2.2 that $\pi_1^+(u)$ is the maximum value that π_1 attains on the level set $H_{j(u)}^{-1}(h(u))$, that is, the largest positive root of $S_{\Gamma(u)}(\pi_1)$ (10). The choice of $\pm$ sign for $x(u)$ determines one of the two families $\xi_\pm(u)$. Note that $\varphi_{X_J}^{\pi}(\xi_\pm(u)) = \varphi_{X_J}^{-\pi}(\xi_\pm(u)) = \xi_\mp(u)$.

Proof. Smoothness follows directly from the fact that π_1^+ is a smooth function of h and j, which in turn are smooth functions of u. Finally p_y is well defined since $\pi_1^+ + j > 0$. The fact that $\xi_\pm(u) \in \mathcal{EM}^{-1}(\Gamma(u))$ can be verified by a direct computation. □

Recall that $\gamma_2(u)$ is a union of $\gamma_2^a(u)$ and $\gamma_2^b(u)$, which are integral curves of Z^u, see (24a) and (24b).

Lemma 12. *For $u \in [0, u_*]$ the $t = \pm 1/2$ end of $\gamma_2^a(u)$ joins the $t = \mp 1/2$ end of $\gamma_2^b(u)$; while for $u \in (u_*, 1]$ both $\gamma_2^a(u)$ and $\gamma_2^b(u)$ are closed curves.*

Proof. For $u \in [0, u_*)$ both $\gamma_2^a(u)$ and $\gamma_2^b(u)$ are not closed curves, because the flow of $Z^u|\mathbf{T}^2_{\Gamma(u)}$ is 2-periodic. Moreover, the end of $\gamma_2^a(u)$ coincides with the start of $\gamma_2^b(u)$ and vice versa (see Fig. 6a). This follows from the fact that

$$\varphi_{Z^u}^1\big(\xi_+(u)\big) = \varphi_{X_J}^{\pi}\big(\varphi_{X_2}^1\big(\xi_+(u)\big)\big) = \varphi_{X_J}^{\pi}\big(\xi_+(u)\big) = \xi_-(u).$$

Consequently, $\varphi_{Z^u}^{1/2}(\xi_+(u)) = \varphi_{Z^u}^{-1/2}(\xi_-(u))$.

For $u \in (u_*, 1]$ the paths $\gamma_2^a(u)$ and $\gamma_2^b(u)$ are closed (see Fig. 6c), because the flow of Z^u is 1-periodic. □

Therefore for $u \in [0, u_*)$ the path $\gamma_2(u)$ is a single closed path formed by joining the end point of $\gamma_2^a(u)$ with the starting point of $\gamma_2^b(u)$ followed by joining the end point of $\gamma_2^b(u)$ with the starting point of $\gamma_2^a(u)$. For $u = u_*$ the path $\gamma_2(u_*)$ is the union of two closed paths with one point $\chi_* = \gamma_2^a(\pm 1/2) = \gamma_2^b(\pm 1/2)$ in common. Thus $\gamma_2(u_*)$ is the figure-8 path shown in Fig. 6b.

We now prove

Lemma 13. *The curves $u \mapsto \gamma_2^a(u)$ and $u \mapsto \gamma_2^b(u)$ are smooth when $u \in [0,1] \setminus \{u_*\}$ and are continuous at $u = u_*$.*

Proof. Recall that $\xi_+(u)$ and $\xi_-(u)$ (23) with $u \in [0,1]$ are smooth curves. For concreteness consider the curve $\gamma_2^a(u)$ which is the integral curve of Z^u with starting point $\xi_+(u) \in \mathcal{EM}^{-1}(\Gamma(u))$. The result then follows using the smoothness of $\xi_+(u)$ and the fact that the vector field Z^u depends smoothly on u when $u \neq u_*$ and continuously at $u = u_*$. Moreover, for $u \in [0,1] \setminus \{u_*\}$, the flow of Z^u on $\mathcal{EM}^{-1}(\Gamma(u)) \cap \mathcal{EM}^{-1}(\mathcal{R}_{\text{reg}})$ is defined for time $[-1/2, 1/2]$. □

Using Lemma 13 we obtain

Lemma 14. *The mapping* $u \mapsto \gamma_2(u)$ *is continuous when* $u \in [0, 1]$.

Proof. By definition of $\gamma_2(u)$

$$\lim_{u \to u_*^\pm} \gamma_2(u) = \lim_{u \to u_*^\pm} \big(\gamma_2^a(u) \cup \gamma_2^b(u)\big) = \lim_{u \to u_*^\pm} \gamma_2^a(u) \cup \lim_{u \to u_*^\pm} \gamma_2^b(u).$$

Using

$$\lim_{u \to u_*^+} \gamma_2^k(u) = \lim_{u \to u_*^-} \gamma_2^k(u) = \gamma_2^k(u_*), \quad k = a, b,$$

which follows from Lemma 13, we obtain

$$\lim_{u \to u_*^+} \gamma_2(u) = \lim_{u \to u_*^-} \gamma_2(u) = \gamma_2(u_*) = \gamma_2^a(u_*) \cup \gamma_2^b(u_*) \qquad \square$$

It is now straightforward to compute the variation of the homology classes $[\gamma_1(u)]$ and $[\gamma_2(u)]$ along the oriented smooth closed path Γ.

Since $[\gamma_1(u)]$ does not vary along Γ, it follows that $[\gamma_1(0)] = [\gamma_1(1)]$.

For every $u \in [0, u_*) \cup (u_*, 1)$ we express the homology class $[\gamma_2(u)]$ in terms of the basis of $\mathrm{H}_1(\mathbf{T}^2_{\Gamma(u)}, \mathbf{Z})$ given by the homology classes $[\beta_1(u)]$ and $[\beta_2(u)]$.

Lemma 15. *For* $u \in [0, u_*)$, *we have* $[\gamma_2(u)] = [\beta_1(u)] + 2[\beta_2(u)]$; *while for* $u \in (u_*, 1]$, *we have* $[\gamma_2(u)] = 2[\beta_2(u)]$. *Moreover,* $u \in [0, u_*) \cup (u_*, 1]$, $[\gamma_1(u)]$ *and* $[\gamma_2(u)]$ *generate the index-*2 *subgroup* $\mathcal{H}_1(\mathbf{T}^2_{\Gamma(u)})$.

Proof. For $u \in [0, u_*)$ recall that

$$\gamma_2(u) : [0, 2] \to \mathcal{EM}^{-1}\big(\Gamma(u)\big) : t \mapsto \varphi^t_{Z^u}\big(\xi_+(u)\big) = \varphi^{\pi t}_{X_J} \circ \varphi^t_{X_2}\big(\xi_+(u)\big).$$

Therefore

$$\begin{aligned}\big[\gamma_2(u)\big] &= \big[[0, 2] \to \mathcal{EM}^{-1}\big(\Gamma(u)\big) : t \mapsto \varphi^{\pi t}_{X_J}(\xi_+)\big] \\ &\quad + \big[[0, 2] \to \mathcal{EM}^{-1}\big(\Gamma(u)\big) : t \mapsto \varphi^t_{X_2}(\xi_+)\big] \\ &= \big[\beta_1(u)\big] + 2\big[\beta_2(u)\big].\end{aligned}$$

The statement for $u \in (u_*, 1)$ follows directly from the definition of $\beta_2(u)$. Taking the limit as $u \to 1^-$ gives the result for $u = 1$.

The homology classes $[\gamma_1(u)] = [\beta_1(u)]$ and $[\gamma_2(u)] = [\beta_1(u)] + 2[\beta_2(u)]$ generate a subgroup of $\mathrm{H}_1(\mathbf{T}^2_{\Gamma(u)}, \mathbf{Z})$ that we denote $\mathcal{H}_1(\mathbf{T}^2_{\Gamma(u)})$. Since $\mathcal{H}_1(\mathbf{T}^2_{\Gamma(u)})$ is also generated by $[\beta_1(u)]$ and $2[\beta_2(u)]$, it follows that $\mathcal{H}_1(\mathbf{T}^2_{\Gamma(u)})$ is an index-2 subgroup of $\mathrm{H}_1(\mathbf{T}^2_{\Gamma(u)}, \mathbf{Z})$. $\square$

This completes the proof of Theorem 1.

4. Discussion

One might get the impression from [9] and [10] and more recently [12] that fractional monodromy is limited to integrable $m_1\colon -m_2$ resonant oscillators, where $m_i \in \mathbf{Z}_{\geqslant 0}$, especially to the 1:−2 case. However, a careful look at the proof in these papers suggests that there should be a more general formulation in line with the proof of the existence of ordinary monodromy given in [4], where no Hamiltonian structure was needed. Obviously any extension beyond concrete examples should give the true category for the structural stability of fractional monodromy.

A possible geometric framework for a general approach to fractional monodromy can be found in [10]. There in Sections 3 and 4 it is shown that the fractional monodromy of the integrable foliation of a concrete integrable 1:−2 resonant oscillator persists if suitable compactifying terms are used to ensure that the integral mapping F is proper with connected fibers. In Section 2 the notion of a weak singularity of $F^{-1}(c)$, $c \in \mathcal{C}$, is defined. This is done by placing $F^{-1}(c)$ in a family of regular fibers of F, defining the transport of chains in this family across $F^{-1}(c)$, classifying the chains which are passable, and finally relating the collection of passable chains to a subgroup of the fundamental group of a regular fiber near $F^{-1}(c)$. For a weakly singular fiber this subgroup has the same rank as the abelianization of the whole group. Giving a topological formulation of a monodromy theorem for generalized monodromy remains a problem for future research. As in the usual monodromy theorem [5], it is desirable to give an explicit relation between the topology of a weakly singular fiber $F^{-1}(\Gamma(u_*))$ of the singular fibration $F^{-1}(\Gamma) \to \Gamma$ over $u_* \in \mathcal{C} \cap \Gamma$ and the generalized monodromy matrix.

For a meaningful generalization of our proof of the existence of fractional monodromy we need more. Our proof relies on a subtle analytic property of the vector fields and the corresponding integral map F defined in (2) that results in the possibility to have well defined limits of the rotation angle Θ (see Lemma 3) at the curve $\mathcal{C}$ of weakly singular critical values of F if we use the modified first integrals (J, H) which define the integral map $\mathcal{EM}$. In other words, the singularity of the curled torus $\Lambda_* = \mathcal{EM}^{-1}(c)$ with $c \in \mathcal{C}$ is so weak, that we can still take a pair of normal vector fields in order to construct cycles on Λ_*. This property raises several questions. Is it necessary for the existence of fractional monodromy in general? In other words, if the set of passable cycles is big enough to form a finite index subgroup of the fundamental group of $\mathbf{T}^2_f = F^{-1}(f)$ for some regular value $f \in \Gamma$, then can we always choose integrals defining the same fibration as F so that the rotation angle has finite limits on $\mathcal{C}$? This question has to be answered before our technique for proving the existence of fractional monodromy can be generalized. In particular it is interesting to analyze further the relation of the choice of (J, H) and the condition $\frac{dh}{dj}|\mathcal{C} = 0$ mentioned in Remark 2. Our system (2) represents a class of systems which satisfy such condition. It is possible to show that under certain fairly general assumptions, this condition is both necessary and sufficient for the existence of the limits of Θ on $\mathcal{C}$. It remains to relate these assumptions to the definition of the fractional monodromy in [10].

Our paper is essentially the initial step in the general program outlined above. It provides an analytical technology to construct a generalized monodromy map across weak singularities of certain integrable fibrations and to define a monodromy subgroup $\mathcal{H}_1$ of $\mathrm{H}_1(\mathbf{T}^2_f)$. As in the case of usual monodromy we can study generalized monodromy using a subgroup of the fundamental group $\pi_1(\mathbf{T}^2_f)$, the subgroup $\mathcal{H}_1$ of $\mathrm{H}_1(\mathbf{T}^2_f)$, or a sublattice pl of the period lattice PL. The first approach was used in [10]. We used $\mathcal{H}_1$. Below we comment on the period lattice approach.

On a regular torus $\mathbf{T}^2_f$ the period lattice $\mathrm{PL}(\mathbf{T}^2_f)$ is the lattice of points $(T_1, T_2) \in \mathbf{R}^2$ such that

$$\varphi^{T_1}_{X_J} \varphi^{T_2}_{X_H}(\xi) = \xi, \quad \text{for all } \xi \in \mathbf{T}^2_f.$$

From the construction of the second action X_{I_2} (5b), it follows that on $\mathbf{T}^2_f$ the period lattice $\mathrm{PL}(\mathbf{T}^2_f)$ is spanned by the vectors $(2\pi, 0)$ and $(-\Theta(f), T(f))$. Because $T(f) \to \infty$ as $f \to \mathcal{C}$, this PL is not defined on curled tori. After rescaling time, the modified period lattice $\mathfrak{PL}(\mathbf{T}^2_f)$ is defined as the lattice of points $(T_1, T_2) \in \mathbf{R}^2$ such that

$$\varphi^{T_1}_{X_J} \varphi^{T_2}_{X}(\xi) = \xi, \quad \text{for all } \xi \in \mathbf{T}^2_f.$$

This lattice is spanned by $(2\pi, 0)$ and $(-\widehat{\Theta}(f), \tau(f))$. Using the rotation number ϑ (21*), we can introduce the continuous period lattice $\mathrm{PL}(u)$ spanned by $(2\pi, 0)$ and $(-\vartheta(u), \tau(u))$. This period lattice is continuous at u_* but $\mathrm{PL}(0) \neq \mathrm{PL}(1)$. The lattice $\mathrm{PL}(u)$ has a sublattice $\mathrm{pl}(u)$ spanned by $(2\pi, 0)$ and $(-2\vartheta(u), 2\tau(u))$, which is continuous at u_* and for which $\mathrm{pl}(0) = \mathrm{pl}(1)$. The sublattice $\mathrm{pl}(u)$, $u \neq u_*$ corresponds to the index-2 subgroup $\mathcal{H}_1(\mathbf{T}^2_{\Gamma(u)})$.

We note that the method we used to compute generalized monodromy does not need a continuous deformation of $u \mapsto \mathcal{H}_1(u)$ along Γ and in particular across the weakly singular critical value $\Gamma(u_*)$. Instead, similar to [9,10], we continue individual elements of $\mathcal{H}_1(0)$ (homology classes) and 'reassemble' them into $\mathcal{H}_1(1)$ at the end point $\Gamma(1) = \Gamma(0)$. In the general situation, it is a legitimate question to ask how $\mathcal{H}_1$ can be continued along Γ. In particular, what is the relation of the homology group $\mathrm{H}_1(\mathcal{EM}^{-1}(\Gamma(u_*)), \mathbf{Z})$ of the weakly singular curled torus at $h = 0$ to the two limits of $\mathcal{H}_1$ of a regular torus as $h \to 0_+$ and $h \to 0_-$ and how they correspond to the limits of pl? Equivalently, we may ask which subgroups of the nonabelian fundamental group of $\mathcal{EM}^{-1}(\Gamma(u_*))$ are involved in this continuation. Recently, these questions have been addressed in [7]. The analytic methods of our paper can be used to give an explicit description of the groups involved.

Following a suggestion of the referee, the bundle of the homology subgroups over Γ *and* the connection on this bundle can be realized as follows. For $f \in \mathcal{R}$ sufficiently close to the given point $c \in \mathcal{C}$, that is, for $f = \Gamma(u)$ and $u \in [u_-, u_+] \setminus \{u_*\}$ in Section 3, our construction of cycles γ gives a continuous surjective mapping $\phi = \phi_f$ from $\mathbf{T}^2_f$ onto Λ_*, which is a diffeomorphism on the complement of the singular circle in Λ_* and is a smooth two-fold covering over the circle. This mapping induces a homomorphism of $\mathrm{PL}(\mathbf{T}^2_f)$ onto $\mathfrak{PL}(\Lambda_*)$ and a corresponding homomorphism μ_f of $\mathrm{H}_1(\mathbf{T}^2_f, \mathbf{Z})$ to $\mathrm{H}_1(\Lambda_*, \mathbf{Z})$. Note that the map μ_f does not depend on the choice of the vector fields (X_1, X_2). One can conjecture that μ_f is injective. Consider the intersection

$$\mu_{\Gamma(u_-)}\big(\mathrm{H}_1\big(\mathbf{T}^2_{\Gamma(u_-)}, \mathbf{Z}\big)\big) \cap \mu_{\Gamma(u_+)}\big(\mathrm{H}_1\big(\mathbf{T}^2_{\Gamma(u_+)}, \mathbf{Z}\big)\big) \cap \mathcal{H}_1(\Lambda_*, \mathbf{Z}) = \mathcal{H}^*_1.$$

Using the results in Section 3, we see that the preimages

$$\mathcal{H}^-_1 = \mu^{-1}_{\Gamma(u_-)}\big(\mathcal{H}^*_1\big) \quad \text{and} \quad \mathcal{H}^+_1 = \mu^{-1}_{\Gamma(u_+)}\big(\mathcal{H}^*_1\big)$$

should be index-2 subgroups of $\mathbf{Z}^2$ lattices $\mathrm{H}_1(\mathbf{T}^2_{\Gamma(u_-)}, \mathbf{Z})$ and $\mathrm{H}_1(\mathbf{T}^2_{\Gamma(u_+)}, \mathbf{Z})$, respectively. Providing that these latter statements are valid, we obtain the isomorphism

$$\mu^{-1}_{\Gamma(u_+)} \circ \mu_{\Gamma(u_-)} : \mathcal{H}^-_1 \mapsto \mathcal{H}^+_1,$$

which can be used to define a continuous connection at $c = \Gamma(u_*)$ but only for 'passable' index-2 subgroups of H_1. Connecting $\mathcal{H}^-_1$ and $\mathcal{H}^+_1$ back to the original sublattice $\mathcal{H}^{(0)}_1 = \mathcal{H}_1(\mathbf{T}^2_{\Gamma(0)}, \mathbf{Z})$ and forward to the final sublattice $\mathcal{H}^{(1)}_1 = \mathcal{H}_1(\mathbf{T}^2_{\Gamma(1)}, \mathbf{Z})$, both of which lie over the point

$\Gamma(0) = \Gamma(1)$, is trivial since the H_1 bundles over $[0, u_-]$ and $[u_+, 1]$ are trivial. Then we can define monodromy in the usual sense of Duistermaat (see Section 1.1) for the bundle of index-2 subgroups $\mathcal{H}_1$ over Γ.

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Appendix A. Reduction of the $\mathbf{S}^1$ symmetry

Since the $\mathbf{S}^1$ action (4) of the flow φ_J of X_J is not free, we need *singular reduction* [3, Appendix B] to remove the $\mathbf{S}^1$ symmetry (4) from the Hamiltonian system $(H, \mathbf{R}^4, \omega)$. We use invariant theory, cf. Chapter I.5 of [3]. In the lemmas below we summarize the information about the reduced system we require in this paper.

Lemma 16. *The algebra of φ_J-invariant polynomials is generated by*

$$J(\xi) = \frac{1}{2}\left(x^2 + p_x^2\right) - \left(y^2 + p_y^2\right), \tag{A.1a}$$

$$\pi_1(\xi) = \frac{1}{2}\left(x^2 + p_x^2\right) + \left(y^2 + p_y^2\right), \tag{A.1b}$$

$$\pi_2(\xi) = \sqrt{2}\left(\left(x^2 - p_x^2\right)y - 2xp_x p_y\right), \tag{A.1c}$$

$$\pi_3(\xi) = \sqrt{2}\left(\left(x^2 - p_x^2\right)p_y + 2xyp_x\right) \tag{A.1d}$$

subject to the relations

$$\Phi = -2\left(\pi_2^2 + \pi_3^2 - (\pi_1 - J)(\pi_1 + J)^2\right) = 0, \quad \text{and} \quad \pi_1 \geqslant |J|. \tag{A.2}$$

Lemma 17. *The reduced phase space P_j is defined by*

$$\pi_2^2 + \pi_3^2 = (\pi_1 - j)(\pi_1 + j)^2, \quad \pi_1 \geqslant |j|, \tag{A.3}$$

as two-dimensional semi-algebraic variety embedded in the ambient space $\mathbf{R}^3$ with coordinates (π_1, π_2, π_3), see Fig. A.1. *The space P_j has the following topology*: *when $j > 0$, P_j is diffeomorphic to $\mathbf{R}^2$*; *when $j \leqslant 0$, it is homeomorphic, but not diffeomorphic, to $\mathbf{R}^2$, having a cusplike singularity at $(\pi_1 = |j|, \pi_2 = \pi_3 = 0)$, when $j = 0$, and a conical singularity when $j < 0$.*

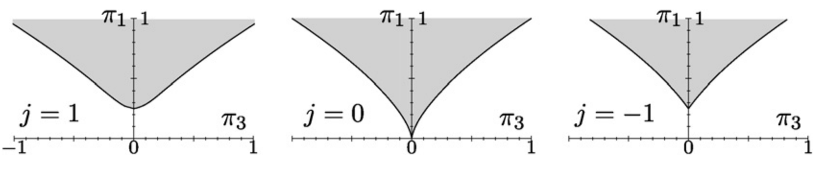


Fig. A.1. Projections V_j of reduced phase spaces P_j on the (π_1, π_3) plane along the π_2 axis. Every point in the boundary ∂V_j (solid line) lifts to one point in P_j with $\pi_2 = 0$; points in the shaded area correspond to two points in P_j.

Proof. Equation (A.2) defines the reduced phase space P_j as a surface in the ambient space $\mathbf{R}^3$ with coordinates $\pi = (\pi_1, \pi_2, \pi_3)$. From (A.3) we see that P_j is a surface of revolution about the axis π_1. Therefore, it is sufficient to study a projection of P_j in a plane containing this axis, see Fig. A.1. The projection map

$$\mathbf{R}^3 \to \mathbf{R}^2 : (\pi_1, \pi_2, \pi_3) \mapsto (\pi_1, \pi_3)$$

is most convenient, because it reduces the extra $\mathbf{Z}_2$ symmetry of the Hamiltonian H_j in (A.4). The image of P_j under this map is called the *fully reduced space* $V_j = P_j/\mathbf{Z}_2$. Points in the boundary $\partial V_j = P_j \cap \{\pi_3 = 0\}$ lift to one point in P_j, other points of V_j lift to two points. We obtain the form of P_j near the lowest point $(|j|, 0, 0)$ by looking at the form of the boundary ∂V_j at the point $(|j|, 0)$. The other assertions are straightforward to prove. □

Since H in (2b) is invariant under the flow φ_J, it induces on P_j a smooth function

$$H_j : P_j \subseteq \mathbf{R}^3 \mapsto \mathbf{R} : \pi = (\pi_1, \pi_2, \pi_3) \mapsto H_j(\pi) = \pi_3 + \epsilon\left(\pi_1^2 - j^2\right), \tag{A.4}$$

called the *reduced Hamiltonian*.

Lemma 18. *The space of smooth functions on P_j, given by restricting smooth functions on $\mathbf{R}^3$ to P_j, forms a Poisson algebra with Poisson bracket*

$$\{\pi_1, \pi_2\} = \frac{\partial \Phi}{\partial \pi_3} = -4\pi_3,$$
$$\{\pi_2, \pi_3\} = \frac{\partial \Phi}{\partial \pi_1} = 2(\pi_1 + j)(3\pi_1 - j),$$
$$\{\pi_3, \pi_1\} = \frac{\partial \Phi}{\partial \pi_2} = -4\pi_2.$$

Proof. Compute the Poisson bracket of π_i and π_j as functions on $\mathbf{R}^4$, express the result in terms of π_1, π_2, π_3, j using (A.1). (This is always possible since $\{\pi_i, \pi_j\}$ is φ_J invariant.) Finally restrict the result to P_j. Note that Φ in (A.2) is a Casimir in this Poisson algebra. □

The reduced Hamiltonian system $(H_j, P_j, \{,\})$ has one degree of freedom. The equations of motion on P_j for the reduced Hamiltonian H_j (A.4) are

$$\dot{\pi}_1 = \{\pi_1, H_j\} = 4\pi_2, \tag{A.5a}$$
$$\dot{\pi}_2 = \{\pi_2, H_j\} = 2(\pi_1 + j)(3\pi_1 - j) + 8\epsilon\, \pi_1\pi_3, \tag{A.5b}$$
$$\dot{\pi}_3 = \{\pi_3, H_j\} = -8\epsilon\, \pi_1\pi_2. \tag{A.5c}$$

Appendix B. The discriminant locus of the integral map

Here we describe the discriminant locus Δ of F (2), that is, the *whole* set of critical values of F which lie in the range of the energy momentum mapping $\mathcal{EM}$, see Fig. B.1.

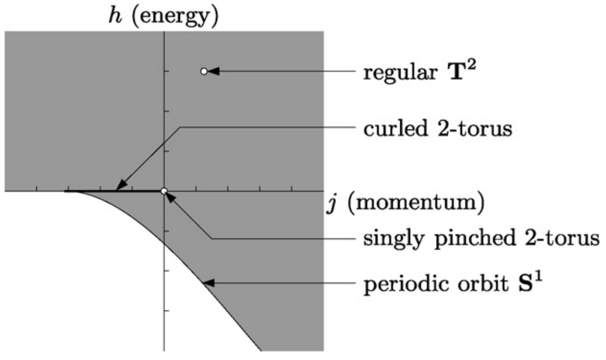


Fig. B.1. The discriminant locus Δ of $\mathcal{EM}$. The range of $\mathcal{EM}$ is shaded.

Lemma 19. *Δ is the union of the image of two curves $\mathcal{C}_1$ and $\mathcal{C}_2$, parameterized by*

$$\mathcal{C}_1 : [0, \infty) \to \mathbf{R}^2 : s \mapsto \big(j(s), h(s)\big) = (-s, 0) \tag{B.1}$$

and

$$\mathcal{C}_2 : \left[\frac{1}{2\epsilon^2}, \infty\right) \to \mathbf{R}^2 : s \mapsto \big(j(s), h(s)\big) = \Big(3s - 8\epsilon^2 s^2 + 4\epsilon s\sqrt{2s(2\epsilon^2 s - 1)},$$
$$- 8s^2\epsilon\big(2s\epsilon^2 - 1\big)\big(8s\epsilon^2 - 3\big) + 4\sqrt{2}\big(s\big(2s\epsilon^2 - 1\big)\big)^{3/2}\big(8s\epsilon^2 - 1\big)\Big) \tag{B.2}$$

which join at the point $P = (-1/(2\epsilon^2), 0)$ in a C^1 but not C^2 fashion. The image of the curve $\mathcal{C}_1|(0, 1/(2\epsilon^2))$ is called the critical line segment $\mathcal{C}$.

Proof. The discriminant locus Δ is the set of points $(j, h) \in \mathcal{EM}(\mathbf{R}^4)$ which are critical values of $\mathcal{EM}$ (2). Writing the Hamiltonian H in terms of invariants of the flow of X_J gives

$$h = \pi_3 + \epsilon\big(\pi_1^2 - j^2\big).$$

Eliminating π_3 from $\pi_2^2 + \pi_3^2 = (\pi_1 - j)(\pi_1 + j)^2$, compare with (A.3), and using the above equation, we obtain

$$\pi_2^2 + p(\pi_1) = 0 \tag{B.3}$$

where

$$p(\pi_1) = \big(h - \epsilon\big(\pi_1^2 - j^2\big)\big)^2 - \big(\pi_1^2 - j^2\big)(\pi_1 + j), \quad \pi_1 \geqslant |j|. \tag{B.4}$$

$(j, h) \in \Delta$ if and only if the polynomial $\pi_2^2 + p(\pi_1)$ has a multiple root, that is, $\pi_2 = 0$ and p (B.4) has a multiple root in $[|j|, \infty)$. The polynomial p has a multiple root in $[|j|, \infty)$ if and only if we can write

$$p(\pi_1) = (\pi_1 - s)^2\big(\epsilon^2\pi_1^2 + u\pi_1 + v\big), \tag{B.5}$$

for $s \in [|j|, \infty)$ and $u, v \in \mathbf{R}$. Equating coefficients of the same power of π_1 in (B.4) and (B.5) gives

$$\begin{aligned} u - 2s\epsilon^2 &= -1, \\ v - 2su + s^2\epsilon^2 &= -2\epsilon^2 j^2 - 2\epsilon h - j, \\ s(su - 2v) &= j^2, \\ s^2 v &= \left(h + \epsilon j^2\right)^2 + j^3. \end{aligned} \tag{B.6}$$

Eliminating u and v from Eqs. (B.6) gives

$$h^2 + j^3 + js^2 + 2s^3 + 2hj^2\epsilon + 2hs^2\epsilon + j^4\epsilon^2 + 2j^2s^2\epsilon^2 - 3s^4\epsilon^2 = 0, \tag{B.7a}$$

$$j^2 - 2js - 3s^2 - 4\epsilon hs - 4j^2s\epsilon^2 + 4s^3\epsilon^2 = 0. \tag{B.7b}$$

We now show how to parameterize the solution set of (B.7). If $s = 0$ then $j = h = 0$. Suppose $s \neq 0$. Then from (B.7b) we get

$$h = -\frac{1}{4s\epsilon}\left(-j^2 + 2js + 3s^2 + 4j^2s\epsilon^2 - 4s^3\epsilon^2\right). \tag{B.8}$$

Using (B.8) to eliminate h from (B.7a) we find

$$(j + s)^2\left((j - 3s)^2 + 16(j - s)s^2\epsilon^2\right) = 0 \tag{B.9}$$

where $s \geqslant |j|$. We have three cases depending on the discriminant

$$\delta = 128\epsilon^2 s^3\left(2\epsilon^2 s - 1\right)$$

of the quadratic factor in (B.9).

Case 1. When $0 < s < 1/(2\epsilon^2)$ we have $\delta < 0$. In this case (B.9) has only one real linear factor. Hence $j = -s$. From (B.8) we find that $h = 0$. This gives the critical line segment $\mathcal{C}$.

Case 2. When $s = 1/(2\epsilon^2)$, Eq. (B.9) becomes $(1 + 2j\epsilon^2)^4 = 0$ that is, $j = -1/(2\epsilon^2)$. From Eq. (B.8), we obtain $h = 0$. This gives the point P.

Case 3. When $s > 1/(2\epsilon^2)$, we have $\delta > 0$. In this case (B.9) has three real linear factors which give rise to three real solution branches; namely

$$\begin{aligned} j &= -s, \\ h &= 0 \end{aligned} \tag{B.10}$$

or

$$\begin{aligned} j &= s\left(3 - 8\epsilon^2 s + 4\epsilon\sqrt{2s(2\epsilon^2 s - 1)}\right), \\ h &= -8s^2\epsilon\left(2s\epsilon^2 - 1\right)\left(8s\epsilon^2 - 3\right) + 4\sqrt{2}\left(s\left(2s\epsilon^2 - 1\right)\right)^{3/2}\left(8s\epsilon^2 - 1\right) \end{aligned} \tag{B.11}$$

or

$$j = s\big(3 - 8\epsilon^2 s - 4\epsilon\sqrt{2s(2\epsilon^2 s - 1)}\big), \tag{B.12a}$$

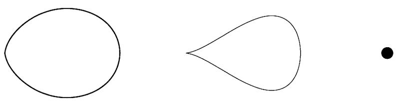
$$h = -8s^2\epsilon\big(2s\epsilon^2 - 1\big)\big(8s\epsilon^2 - 3\big) - 4\sqrt{2}\big(s\big(2s\epsilon^2 - 1\big)\big)^{3/2}\big(8s\epsilon^2 - 1\big). \tag{B.12b}$$

From (B.12a) and the fact that $s > 1/(2\epsilon^2)$ we see that $s < |j|$ which contradicts the demand that s is a double root of p (B.4) in $[|j|, \infty)$. Therefore this case must be excluded. This proves the lemma. □

Appendix C. Fibers of the integral map

To characterize the solutions of the equations of motion (A.5) qualitatively, we determine the topology of the level set of H_j on P_j. Let $\mathcal{R} \subseteq \mathbf{R}^2$ be the image of the energy-momentum map $\mathcal{EM}$, $\mathcal{R}_{\text{reg}} \subseteq \mathcal{R}$ be the set of regular values of $\mathcal{EM}$, and $\mathcal{R}_{\min} \subset \mathcal{R}$ be the set of absolute minimum values of H_j for each j.

Lemma 20. *Suppose that $(j, h) \in \mathcal{R}$. The h-level set of the reduced Hamiltonian H_j* (A.4) *on P_j has one of the following three topological types*:



When $(j, h) \in \mathcal{R}_{\text{reg}}$, then $H_j^{-1}(h)$ is a smooth circle; *when $(j, h) \in \partial\mathcal{R} = \mathcal{C}_2 \cup \mathcal{C}_1 \setminus \mathcal{C}$* (B.2) *with $j \neq 0$, then $H_j^{-1}(h)$ is a point*; *when $j \in (-\frac{1}{2}\epsilon^{-2}, 0)$, that is, $(0, j)$ lies on the critical line segment $\mathcal{C}$, then the level set $H_j^{-1}(0)$ is homeomorphic* (*but* not *diffeomorphic*) *to a circle, having a cusplike singularity at $\pi = (|j|, 0, 0)$.*

The proof of the lemma can be given by studying analytically the intersections $H_j^{-1}(h)$ with V_j and subsequently lifting these intersections to P_j. The case when this intersection is a point has been already studied in Appendix B. In Fig. C.1 it corresponds to the level sets marked a. Furthermore, this kind of intersection can be either a regular point, such as top and bottom left cases in Fig. C.1, or the singular point, bottom right. As shown in Appendix B the latter occurs only when $-\frac{1}{2}\epsilon^{-2} < j < 0$ and the level set corresponds to the absolute minimum value of h.

If the intersection $H_j^{-1}(h) \cap V_j$ is a line segment then its minimum and maximum π_1 value are denoted as π_1^- and π_1^+, respectively, defined in Section 2.2 as the two real roots of $S_{j,h}(\pi_1)$ (10) that are greater than or equal to $|j|$. In the special case of interest in our work, an expansion in h for h close to 0 for these roots is given in Appendix E.2. Notice that π_1^+ is given asymptotically for all j but for π_1^- we can not do this.

From the analytic expression of the lifting map $V_j \to P_j$ we can easily obtain that when $\pi_1^\pm$ are regular points the intersection lifts to a smooth circle. For as if π_1^- is a singular point of V_j and $-\frac{1}{2}\epsilon^{-2} < j \leqslant 0$ the intersection lifts to a singular circle.

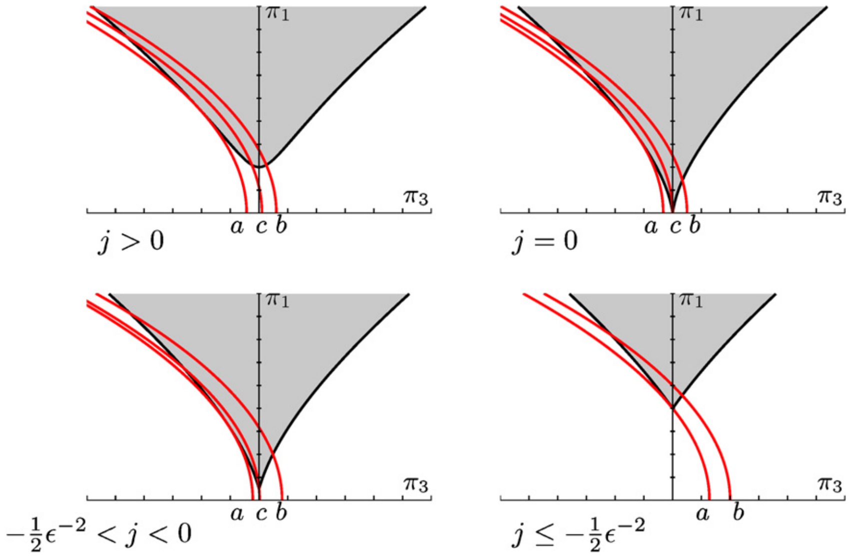


Fig. C.1. Projections of the intersections $P_j \cap \{H_j = h\}$ on the plane $\{\pi_2 = 0\}$. The phase space P_j is shown as in Fig. A.1; the level sets $\{H_j = h\}$ are shown by bold solid curves.

Lemma 21. *The $\mathcal{EM}$ map has four types of fibers $\mathcal{EM}^{-1}(f)$.*

(1) *A regular torus if $f \in \mathcal{R}_{\mathrm{reg}}$.*
(2) *A curled torus if $f \in \mathcal{C}$.*
(3) *A pinched curled torus if $f = (0, 0)$.*
(4) *A periodic orbit of period 2π or π if $f \in \partial\mathcal{R}$.*

Proof. By the Arnol'd–Liouville theorem the regular points of $\mathcal{EM}$ lift to regular 2-tori. Their projections on P_j are smooth circles. The point level sets of H_j on P_j lift to relative equilibria, which are $\mathbf{S}^1$ orbits. If the point Q is a singular point of P_j, then it lifts to a periodic orbit of period π (this happens when $j \leqslant -\frac{1}{2}\epsilon^{-2}$). Otherwise the point lifts to a periodic orbit of period 2π.

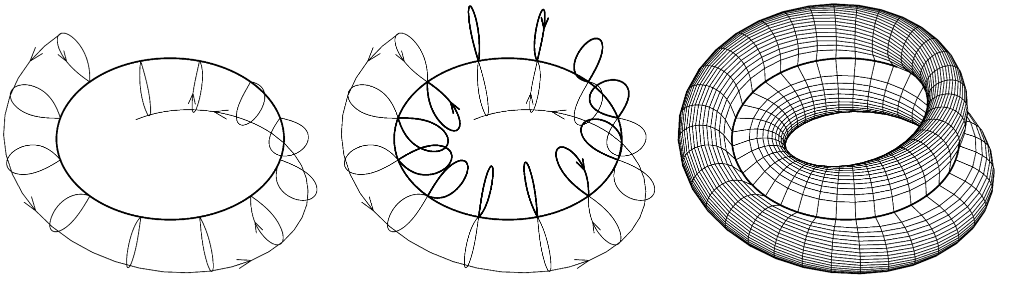


Fig. C.2. Possible reconstruction of the curled torus embedded in $\mathbf{R}^3$ (right) as a 'petal' (leftmost panel) which follows regular 2π-periodic orbits of X_J (fine spiral line with arrows) and 'curls' about the singular π-periodic orbit of X_J (bold circle), and as a figure eight (center panel) which maps to itself with a twist after period π.

When (j,h) lies on the critical line segment $\mathcal{C}$, the singular point Q reconstructs to the non-degenerate critical manifold $\mathcal{M}$, which is given by

$$\{x = p_x = 0,\ p_y^2 + y^2 = |j|\}.$$

In fact $\mathcal{M}$ is the image of a periodic orbit $\gamma_{h,j}$ of X_H with primitive period π. Since the linear Poincaré map of $\gamma_{j,h}$ is $\varphi^{\pi}_{X_J} = -\mathrm{id}$, the orbit $\gamma_{j,h}$ is hyperbolic with reflection. So $H_j^{-1}(h)$ reconstructs to $\gamma_{j,h}$ together with its stable and unstable manifolds. The latter are twisted, but $\mathcal{EM}^{-1}(j,h)$ is orientable. In other words, for (j,h) on the critical line segment $\mathcal{C}$, the (j,h)-level set of $\mathcal{EM}$ is a *curled* 2*-torus*; namely, a cylinder on a figure eight whose ends are identified after performing a half twist, see Fig. C.2. Here the image of $\gamma_{j,h}$ is the curve formed from the crossing point of the figure eight after making the identification. When $j = h = 0$, the curve $\gamma_{j,h}$ collapses to a point and $\mathcal{EM}^{-1}(j,h)$ is a *pinched curled* 2*-torus*. □

Appendix D. Relation of $\mathcal{EM}$ to the integral map of [9]

The integral map $F = (F_1, F_2)$ defined in [9] is

$$F_1(\xi) = \frac{1}{2}\big(x^2 + p_x^2\big) - \big(y^2 + p_y^2\big), \tag{D.1a}$$

$$F_2(\xi) = \sqrt{2}\big(\big(x^2 - p_x^2\big)p_y + 2xyp_x\big) + \epsilon\left(\frac{1}{2}\big(x^2 + p_x^2\big) + \big(y^2 + p_y^2\big)\right)^2. \tag{D.1b}$$

Notice that $F_1 = J$ and $F_2 = H + \epsilon J^2$.

Lemma 22. *The integral maps* $\mathcal{EM}$ (2) *and* F (D.1) *define the same fibration* $\mathcal{F}$*. Moreover, the image of* $\mathcal{EM}$ *is diffeomorphic to the image of* F*.*

Proof. The map

$$\Psi : \mathbf{R}^2 \to \mathbf{R}^2 : (f_1, f_2) \mapsto \big(f_1, f_2 - \epsilon f_1^2\big) = (j,h)$$

is a diffeomorphism, because $D\Psi(f_1,f_2) = \begin{pmatrix} -2\epsilon f_1 & 1 \\ 1 & 0 \end{pmatrix}$ is invertible and the map $(j,h) \mapsto (f_1,f_2) = (j, h + \epsilon j^2)$ is Ψ^{-1}. Since $\mathcal{EM}^{-1}(\Psi(f_1,f_2)) = F^{-1}(f_1,f_2)$, the level sets of $\mathcal{EM}$ define the same integrable fibration as the level sets of F. □

The reason we have chosen $\mathcal{EM}$ (2) instead of F (D.1) is that the rotation number of X_{F_2} with respect to $X_{F_1} = X_J$ becomes infinite as we approach the critical line segment. In particular let $\Phi(f_1,f_2)$ be the rotation number on the torus $F^{-1}(f_1,f_2) = \mathcal{EM}^{-1}(\Psi(f_1,f_2))$. Then a simple calculation along the lines of that given in Section 2.2 shows that

$$\Phi(f_1,f_2) = \Theta\big(\Psi^{-1}(f_1,f_2)\big) + 2f_1 T\big(\Psi^{-1}(f_1,f_2)\big). \tag{D.2}$$

This means that by using $H = F_2 - \epsilon F_1^2$ instead of F_2 we have essentially removed from the rotation number Φ the singular behavior of the first return time near the critical line segment and we have obtained a finite rotation number Θ.

Appendix E. Asymptotic expansions of dynamical quantities near $\mathcal{C}$

In this section we find asymptotic expansions for the first return time $T(j,h)$, the modified first return time $\tau(j,h)$ and the rotation number $\Theta(j,h)$ near the critical line, that is, for $j<0$ and h close to 0. Recall that $T(j,h)$ (11) and $\Theta(j,h)$ (14) are integrals of the form

$$\int_{\pi_1^-}^{\pi_1^+} f(\pi_1)\frac{1}{\sqrt{S_{j,h}(\pi_1)}}\,\mathrm{d}\pi_1, \tag{E.1}$$

where $f(\pi_1)=1/2$ in the case of $T(j,h)$ and $h/(j+\pi_1)$ in the case of $\Theta(j,h)$. Considerations similar to those of Section 2.2 show that the modified first return time $\tau(j,h)$ is given also by (E.1) with $f(\pi_1)=(j+\pi_1)/2$.

In order to compute these integrals we define $u=\pi_1+j$. Then (E.1) becomes

$$\int_{u^-}^{u^+} f(u-j)\frac{1}{\sqrt{Q_{j,h}(u)}}\,\mathrm{d}u, \tag{E.2}$$

where $u^\pm=\pi_1^\pm+j$ and $Q_{j,h}(u)=P_{j,h}(u-j)$. Explicitly we have

$$T(j,h)=\frac{1}{2}\int_{u^-}^{u^+}\frac{1}{\sqrt{Q_{j,h}(u)}}\,\mathrm{d}u, \tag{E.3}$$

$$\Theta(j,h)=h\int_{u^-}^{u^+}\frac{1}{u\sqrt{Q_{j,h}(u)}}\,\mathrm{d}u, \tag{E.4}$$

and

$$\tau(j,h)=\frac{1}{2}\int_{u^-}^{u^+}\frac{u}{\sqrt{Q_{j,h}(u)}}\,\mathrm{d}u. \tag{E.5}$$

Each such quantity is an elliptic integral which we compute in terms of the roots of the polynomial $Q_{j,h}(u)$ and complete elliptic integrals of the first and third kind.

E.1. Asymptotic expansions for complete elliptic integrals

The complete elliptic integral of the first kind is

$$K(m)=\int_0^{\pi/2}\frac{1}{\sqrt{1-m\sin^2 t}}\,\mathrm{d}t=\frac{1}{2}\int_0^1\frac{1}{\sqrt{y(1-y)(1-my)}}\,\mathrm{d}y \tag{E.6}$$

and that of the third kind is

$$\Pi(n,m) = \int_0^{\pi/2} \frac{1}{1-n\sin^2 t}\frac{1}{\sqrt{1-m\sin^2 t}}\,\mathrm{d}t$$
$$= \frac{1}{2}\int_0^1 \frac{1}{(1-ny)\sqrt{y(1-y)(1-my)}}\,\mathrm{d}y, \tag{E.7}$$

We give the following asymptotic expansions for these integrals without proof.

Lemma 23. *For $x > 0$ close to 0 we have*

$$K\bigl(1-x+O(x^2)\bigr) = 2\log 2 - \frac{1}{2}\log x + O(x). \tag{E.8}$$

Lemma 24. *For $x > 0$ close to 0 and $k_2 > k_1 > 0$ we have*

$$\Pi\bigl(1-k_1x+O(x^2), 1-k_2x+O(x^2)\bigr) = -\frac{1}{4}\log x - \frac{\tan^{-1}\sqrt{\frac{k_2}{k_1}-1}}{k_1\sqrt{\frac{k_2}{k_1}-1}}\frac{1}{x} + O(1). \tag{E.9}$$

Lemma 25. *For $x > 0$ close to 0 and $n < 0$ we have*

$$\Pi\bigl(n+O(x), 1-x+O(x^2)\bigr)$$
$$= \frac{1}{2(1-n)}\bigl(-\log x + 4\log 2 + 2\sqrt{-n}\tan^{-1}\sqrt{-n}\bigr) + O(x). \tag{E.10}$$

E.2. The roots of $Q_{j,h}(u)$

It is easy to show that near the critical line segment $\mathcal{C}$, that is, for h close to 0 and $-\frac{1}{2}\epsilon^{-2} < j < 0$, $Q_{j,h}(u)$ has four real roots. For $h = 0$ two of the roots coincide. Specifically, for $h = 0$ we have

$$Q_{j,h}(u) = u^2(u-2j)\bigl(1-\epsilon^2(u-2j)\bigr), \tag{E.11}$$

with roots $w_1^{(0)} = 2j$, $w_2^{(0)} = w_3^{(0)} = 0$ and $w_4^{(0)} = 2j + \epsilon^{-2}$.

For h close to 0 we expand the roots of $Q_{j,h}(u)$ in a power series in h, namely,

$$w_i = w_i^{(0)} + w_i^{(1)}h + w_i^{(2)}h^2 + \cdots, \quad i = 1, \ldots, 4, \tag{E.12}$$

where

$$w_1 = 2j + O\left(h^2\right), \tag{E.13a}$$

$$w_2 = -\frac{1}{\sqrt{-2j + 2\epsilon j}} h + O\left(h^2\right), \tag{E.13b}$$

$$w_3 = \frac{1}{\sqrt{-2j - 2\epsilon j}} h + O\left(h^2\right), \tag{E.13c}$$

$$w_4 = 2j + \frac{1}{\epsilon^2} + \frac{2\epsilon}{1 + 2\epsilon^2 j} h + O\left(h^2\right). \tag{E.13d}$$

In the following sections we will refer to the roots of $Q_{j,h}(u)$ sorted in ascending order. Notice that for $h > 0$ we have $w_1 < w_2 < w_3 < w_4$ while for $h < 0$ we have $w_2 > w_3$. Therefore for $h > 0$ we define $u_1 = w_1$, $u_2 = w_2$, $u_3 = w_3$ and $u_4 = w_4$ while for $h < 0$ we define $u_1 = w_1$, $u_2 = w_3$, $u_3 = w_2$ and $u_4 = w_4$. In this way we have $u_1 < u_2 < u_3 < u_4$ for all small $h \neq 0$.

E.3. The first return time near $\mathcal{C}$

Recall that the first return time is given by the integral

$$T(j,h) = \frac{1}{2} \int_{u_-}^{u_+} \frac{1}{\sqrt{Q_{j,h}(u)}} \,\mathrm{d}u. \tag{E.14}$$

Close to the critical line segment we can express $Q_{j,h}(u)$ in the form

$$Q_{j,h}(u) = -\epsilon^2 (u - u_1)(u - u_2)(u - u_3)(u - u_4) \tag{E.15}$$

where $u_i \in \mathbf{R}$ are given by the series (E.13). Notice that $u_- = u_3$ and $u_+ = u_4$ and

$$T(j,h) = \frac{1}{2\epsilon} \int_{u_3}^{u_4} \frac{1}{\sqrt{-(u - u_1)(u - u_2)(u - u_3)(u - u_4)}} \,\mathrm{d}u. \tag{E.16}$$

In order to compute this integral (and those in the next section) we make the transformation

$$y = \frac{(u - u_3)(u_4 - u_2)}{(u - u_2)(u_4 - u_3)} \tag{E.17}$$

and we define

$$m = \frac{(u_2 - u_1)(u_4 - u_3)}{(u_3 - u_1)(u_4 - u_2)}. \tag{E.18}$$

Then $T(j,h)$ becomes

$$T(j,h) = \frac{1}{2\epsilon\sqrt{(u_3-u_1)(u_4-u_2)}} \int_0^1 \frac{1}{\sqrt{y(1-y)(1-my)}}\,dy$$
$$= \frac{1}{\epsilon\sqrt{(u_3-u_1)(u_4-u_2)}} K(m) \tag{E.19}$$

where $K(m)$ is the complete elliptic integral of the first kind. Notice that using (E.13) we obtain that

$$m = 1 - \frac{1}{\sqrt{2}(-j)^{3/2}(1+2\epsilon^2 j)^2}|h| + O\big(h^2\big). \tag{E.20}$$

Therefore from Lemma 23 we obtain

$$K(m) = -\frac{1}{2}\log|h| + O(1). \tag{E.21}$$

We also have

$$\frac{1}{\sqrt{(u_3-u_1)(u_4-u_2)}} = \frac{\epsilon}{\sqrt{-2j}\sqrt{2\epsilon^2 j+1}} + O(h). \tag{E.22}$$

Combining Eqs. (E.21) and (E.22) we obtain that for h close to 0,

$$T(j,h) = -\frac{1}{2\sqrt{-2j}\sqrt{2\epsilon^2 j+1}}\log|h| + O(1). \tag{E.23}$$

This shows immediately that $\lim_{h\to 0} T(j,h) = \infty$ when $j < 0$.

E.4. The modified first return time near $\mathcal{C}$

Recall that the modified first return time is given by the integral

$$\tau(j,h) = \frac{1}{2}\int_{u_-}^{u_+} \frac{u}{\sqrt{Q_{j,h}(u)}}\,du. \tag{E.24}$$

We make the transformation (E.17) and obtain

$$\tau(j,h) = \frac{1}{\epsilon\sqrt{(u_3-u_1)(u_4-u_2)}}\big(u_2 K(m) + (u_3-u_2)\Pi(n,m)\big) \tag{E.25}$$

where m is given by (E.20) and

$$n = \frac{u_3-u_4}{u_2-u_4} = 1 - \frac{\sqrt{2}\epsilon^2}{\sqrt{-j}(1-2j\epsilon^2)^2}|h| + O\big(h^2\big). \tag{E.26}$$

Combining Lemmas 23 and 24 with the asymptotic expressions (E.13) we obtain from (E.25) that for h close to 0,

$$\tau(j,h) = \frac{1}{\epsilon}\tan^{-1}\left(1 + \frac{1}{2\epsilon^2 j}\right) + O(h). \tag{E.27}$$

E.5. The rotation angle near $\mathcal{C}$

Recall that the rotation angle is given by the integral

$$\Theta(j,h) = \frac{h}{2}\int_{u_-}^{u_+} \frac{1}{u\sqrt{Q_{j,h}(u)}}\,\mathrm{d}u. \tag{E.28}$$

Again we make the transformation (E.17). We obtain that

$$\Theta(j,h) = \frac{2h}{\epsilon\sqrt{(u_3-u_1)(u_4-u_2)}}\left(\frac{1}{u_2}K(m) + \frac{u_2-u_3}{u_2u_3}\Pi(n,m)\right) \tag{E.29}$$

where

$$n = \frac{u_2(u_3-u_4)}{u_3(u_2-u_4)} = \frac{1+\operatorname{sgn}(h)\epsilon\sqrt{-2j}}{1-\operatorname{sgn}(h)\epsilon\sqrt{-2j}} + O(h) < 0. \tag{E.30}$$

Combining Lemmas 23 and 25 with the relations (E.13) from (E.29) we obtain that for h close to 0 and $j < 0$

$$\Theta(j,h) = -2\tan^{-1}\left(\frac{1+\operatorname{sgn}(h)\epsilon\sqrt{-2j}}{1-\operatorname{sgn}(h)\epsilon\sqrt{-2j}}\right)^{1/2} + O(h). \tag{E.31}$$

Some algebraic manipulation gives

$$\Theta(j,h) = \begin{cases} \frac{\pi}{2} + \sin^{-1}(\epsilon\sqrt{-2j}\,) + O(h), & \text{for } h > 0,\\ -\frac{\pi}{2} + \sin^{-1}(\epsilon\sqrt{-2j}\,) + O(h), & \text{for } h < 0. \end{cases} \tag{E.32}$$

Appendix F. Dynamics on a curled torus

We study the flow of the vector fields X and Z^u on the curled torus $\mathrm{T}_* = \mathcal{EM}^{-1}(f_*) = \mathcal{EM}^{-1}(\Gamma(u_*))$.

Lemma 26. *On* T_* *the points* $\xi_\pm(u_*)$ (23) *have coordinates*

$$\left(\pm\left(2j+\epsilon^{-2}\right)^{1/2}, 0, 0, (\sqrt{2}\,\epsilon)^{-1}\right),$$

where $(j,h) = \Gamma(u_*)$. *The integral curve* $t \mapsto (x_\pm(t), p_x(t), y(t), p_y(t))$ *of* X *is given by*

$$x_{\pm}(t) = \pm\left(2j + \frac{1}{2\epsilon^2}\bigl(1+\cos(4\epsilon t)\bigr)\right)^{1/2}, \tag{F.1a}$$

$$p_x(t) = 0, \tag{F.1b}$$

$$y(t) = \frac{1}{2\sqrt{2}\epsilon}\sin(4\epsilon t), \tag{F.1c}$$

$$p_y(t) = \frac{1}{2\sqrt{2}\epsilon}\bigl(1+\cos(4\epsilon t)\bigr). \tag{F.1d}$$

Proof. The expressions (F.1) satisfy the equation $\dot{\xi}(t) = X(\xi(t))$. A simple calculation shows that for $t = 0$ they have the correct initial condition $\xi_+(u_*)$. □

Notice that the solution (F.1) reaches the singular set $\{x = p_x = 0\}$ for the first positive (or negative time) when $1 + \cos(4\epsilon t) = -4\epsilon^2 j$. We denote this time by $t_* = \frac{1}{2\epsilon}\arccos(\epsilon\sqrt{2|j|})$. Note that $\lim_{u\to u_*}\tau(\Gamma(u_*)) = 2t_*$ from (E.27). Denote $\tau_* = \tau(\Gamma(u_*))$ and $\vartheta_* = \vartheta(u_*) = \pi/2 + \sin^{-1}(\epsilon\sqrt{-2j})$.

Lemma 27. *The integral curve $t \mapsto \zeta_{\pm}(t)$ of Z^u through $\xi_{\pm}(u_*)$ is given by*

$$\zeta_{\pm}(t) = R(-\vartheta_* t)\xi_{\pm}(\tau_* t) \tag{F.2}$$

where $R(t) = \operatorname{diag}(R_t, R_{-2t})$ (4).

Proof. From (22) and $[X_J, X] = 0$ we obtain that the flow $\varphi_{Z^{u_*}}|\mathrm{T}_*$ is

$$\varphi^t_{Z^{u_*}}\big|\mathrm{T}_* = \varphi^{-\vartheta_* t}_{X_J} \circ \varphi^{\tau_* t}_{X}. \tag{F.3}$$

The lemma follows from the analytic expression of the flow φ_{X_J} (4). □

Lemma 28. $\varphi^{\pm 1/2}_{Z^{u_*}}(\xi_+(u_*)) = \varphi^{\pm 1/2}_{Z^{u_*}}(\xi_-(u_*)) = (0, 0, 0, -|j|)$.

Proof. From

$$\xi_{\pm}(\tau_*/2) = \xi_{\pm}(t_*) = \left(0, 0, \sqrt{|j|(1+2\epsilon^2 j)}, -\sqrt{2}\epsilon j\right)$$

we get

$$R(-\vartheta_*/2)\xi_{\pm}(\tau_*/2) = \bigl(0, 0, 0, -|j|\bigr). \qquad \square$$

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